BIOMATHEMATICS

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BEING

THE PRINCIPLES OF MATHEMATICS FOR STUDENTS OF BIOLOGICAL SCIENCE

BY

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INTRODUCTION

BY

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WITH MANY WORKED NUMERICAL EXAMPLES CHOSEN FROM THE DIFFERENT BRANCHES OF BIOLOGY, AND 161 DIAGRAMS

Second Edition, Reset and Enlarged

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AFFECTIONATELY DEDICATED TO

MY DAUGHTER

PATRICIA.

"He who knows mathematics and does not make use of his knowledge, to him applies the verse in Isaiah (v. 12), 'They regard not the work of the Lord, neither consider the operation of His hands.' "

THE TALMUD.

"The laws by which God has thought good to govern the Universe are surely subjects of lofty contemplation; and the study of that symbolical language by which alone these laws can be fully deciphered is well deserving of his [man's] noblest efforts."

PROFESSOR SEDGWICK.

"The living and the dead, things animate and inanimate, we dwellers in this world and this world wherein we dwell, are bound alike by physical and mathematical law."

D'ARCY W. THOMPSON.

"The ultimate aim of embryology is the mathematical derivation of the adult from the distribution of growth in the germ."

WILHELM HIS.

PREFACE TO FIRST EDITION.

In a recent signed article reviewing two books on Biochemistry, the reviewer—one of the foremost of British physiologists—laments the fact that so much of what is evidently important is veiled in mathematical language. He says: "My own acquaintance with Mathematics is not deep enough to be able to criticise, and I fancy the majority of physiologists and students (even honours students) are much in the same predicament." The time is, therefore, ripe for a book which should explain to the biological student those portions of the so-called higher mathematics which are now being utilised in the study and investigation of biological problems. The present work is meant to fill this gap in biological literature.

The book is designed to fulfil a double object. It aims, in the first instance, at affording the reader sufficient mathematical knowledge to follow intelligently the records of the more modern researches in the various fields of biological science. In addition, it is hoped that a mastery of the book will enable the laboratory investigator to make use of the principles of Mathematics for the purpose of co-ordinating his experimental results. From the kindly Introduction by Professor Sir William M. Bayliss, it appears that I have succeeded in achieving both of these aims.

It is a great pleasure to acknowledge here my indebtedness to the following gentlemen for various kindnesses shown to me either in connection with the manuscript or during the progress of the book through the press: Professors Sir William M. Bayliss, D.Sc., LL.D., F.R.S., and H. E. Roaf, M.D., D.Sc., read the manuscript and made helpful suggestions. To Sir William Bayliss I am particularly grateful for honouring me and my book with his learned Introduction. Sir Walter M. Fletcher, M.D., Sc.D., F.R.S., Secretary of the Medical Research Council, Sir Sydney Russell-Wells, M.D., F.R.C.P., M.P., ex-Vice-Chancellor of the University of London, and Mr. Fred S. Spiers, O.B.E., B.Sc., etc., Secretary of the Faraday Society, have taken great trouble with the manuscript. Professor D'Arcy

W. Thompson, C.B., D.Litt., F.R.S., has read the proofs of the earlier portions of the book, and Dr. Major Greenwood and his colleagues in the Statistical Department of the Ministry of Health have taken very great pains in connection with the proofs of the chapter on Biometrics. Dr. W. A. M. Smart, Demonstrator of Physiology at the London Hospital Medical College, supplied me with the proofs of a couple of formulæ; these are acknowledged in the text. Dr. A. G. M'Kendrick, F.R.S. Ed., Superintendent of the Research Laboratories of the Royal College of Physicians of Edinburgh, and especially Sir James Crichton Browne, M.D., D.Sc., LL.D., F.R.S., have shown me other very great kindnesses as the book was passing through the press. Lastly, I wish to pay a tribute of reverence and respect to the memory of my dear friend, Dr. J. W. Ballantyne, who, though not a mathematician, took a great interest in the book.

W. M. FELDMAN.

1923.

PREFACE TO SECOND EDITION.

In preparing this new edition, the book has been subjected to a very close revision. Not only has every page been minutely scrutinised to eliminate any numerical or printer's errors, but nearly every chapter has, it is believed, been improved by the addition of new matter and the deletion of some unimportant matter. In addition, the arrangement of the chapters has in some cases been altered and two new chapters have been added, on Nomography and Estimation of Errors of Observation. The chapter on Biometry has been re-written, as well as considerably enlarged, and it is hoped thereby improved.

I am very greatly indebted to the many readers who have written to me expressing their appreciation of the first edition of the book and who have made useful suggestions for this edition—many of the latter have been adopted. I am particularly grateful to Dr. Ernest Athole Ross, of Winchester, for his unbounded zeal in connection with the detection of the numerous errors and misprints that marred the pages of the first edition. Dr. Ross most diligently studied the book and of his own accord sent me a most imposing list of errors, all of which have now been eliminated.

To the publishers also I owe a great debt of gratitude for the excellent manner in which they have supervised the passage of the book through the press. I especially appreciate the close scrutiny to which they have subjected the various computations to ensure freedom from inaccuracies of calculation and typographical errors.

It is gratifying both to me and to the publishers that a new edition should have been called for within a few years. This is the best evidence that the book has supplied a need. The book has been out of print for some three years, as owing to various circumstances the publication of a new edition had to be delayed. It is hoped now, however, that this new edition will prove of still greater help to those for whom it is intended.

W. M. FELDMAN.

London, W. 1,
November 1934.

INTRODUCTION.*

At the present day it is scarcely necessary to combat the old prejudice that the application of mathematical treatment to the Biological sciences is a serious error. No doubt caution must be used in the process, as will be pointed out below. But in many aspects these sciences have now reached a stage in which the use of mathematics has become not only profitable, but indispensable. So far as I am aware, there is no book in the English language, or for that matter in any other language, which fills the place of this work by Dr. Feldman. Mellor's "*Higher Mathematics*," valuable as it is for the student of physics or chemistry, contains more than the biologist needs as yet, while it omits matter which is of importance for workers in the domain of the phenomena of life. Much more is this the case with the regular text-books of mathematics. The variety of knowledge that the physiologist, for example, has to call to his aid is so vast that he really cannot spare the time to master these text-books. The present work seems to me to have succeeded in giving just what is likely to be useful. It frequently happens that the biologist at the time in his career when the value of mathematical knowledge forces itself upon him finds that he has forgotten much of what he learned when a student, since he has had little or no occasion to use it in the meantime. Even if he has not forgotten his early studies, it is to be feared that but too often he would realise the uselessness of most of them. He has probably been taught too much dull and unedifying trigonometry and scarcely anything of the fascinating problems of the calculus of continuous changes. Dr. Feldman has wisely begun at the beginning. His book will be useful not only to research workers who wish to subject their experimental results to mathematical treatment, but also to those who merely require to be able to understand the expressions given in papers which they read. It is not to be expected that a book of this kind can be grasped completely

* To First Edition (1923).

by reading it through more or less rapidly. Each step should be mastered before the next is taken.

We may, I take it, accept the statement that the ultimate aim of all science is to express in a mathematical form the discoveries that have been made. In that part of biology with which I am most familiar, the experimental investigation of vital processes, the usual course may be described as follows. We first of all find that the presence of some particular phenomenon is always associated with that of some other, so that when we arrange that this latter shall be present, we know that the former will show itself also. It is often said that the latter is the "cause" of the former, although this way of putting it may be philosophically incorrect. Probably all that we are justified in saying is that the presence of the one always involves that of the other. It is in many cases of biological research impossible to proceed further than this first stage, at least up to the present time. But it must not be supposed that discoveries of this merely qualitative kind are of no importance. In most cases, however, a further step may be taken by *measuring* the magnitude of the "effect" in relation to that of the "cause," if we may be allowed to use these terms for convenience. Suppose that we determine the degree of abolition of response to a constant stimulus as this response is diminished by the action of successive known doses of a poison. We thus obtain two sets of numbers, one set being some mathematical function of the other set. The next step is to find out what function this is—in other words, to express the results in a mathematical formula. As would be expected, this is not always a simple matter. Help is often obtained by making a "graph," in which case we make use of the methods of co-ordinate geometry. Inspection of this curve may suggest various formulæ to be tested. In the case mentioned we may find that a curve belonging to the family of parabolas satisfies the data. But if we proceed to affirm that this shows that the phenomenon is one of adsorption, we begin to tread on dangerous ground and much caution is required if we are to make real progress. As Dr. Feldman points out on p. 89,* "occasionally two or more different formulæ will give results all of which are in agreement with observation." He gives cases of this. Indeed, we have to resort to further experiment in order to test whether our interpretation is in accordance with other characteristics of the process assumed in our working hypothesis.

We see one of the uses of the application of mathematics to

* Page 90 this Edition.

our experimental data. In the particular example given, if it had not been for the form of the curve, we might not have suspected adsorption as a controlling factor, and we are therefore led to make the experimental tests necessary to find out whether it is so or not.

Before we leave this example, we may note that the expression for the adsorption isothermal contains two arbitrary constants, which are given appropriate values in any actual case. This fact gives, as is clear, a large degree of flexibility to the equation, and necessitates special caution in drawing conclusions from the fact that it applies to any particular case. The same statement may be made as to the Barcroft-Hill formula for the dissociation of oxy-hæmoglobin. This can be interpreted on the basis of mass action formulæ, but it has been stated that the data can also be fitted by an adsorption formula.

Mathematics is a powerful tool, and may do damage unless well under control. But under such control it will do what no other tool can do. It is in the application of the compound interest law to physiological phenomena, or rather in conclusions drawn from this, that it seems to the writer that unjustifiable statements are often made. As is shown in the text, there are a number of phenomena in which the rate of further change is proportional to the amount of change which has already taken place, or, in other words, to the amount of material actually in process of change at the moment. One of these cases is that of those simple chemical reactions in which the rate of change is proportional to the concentration of one of the reacting molecules. This is a direct result of the law of mass action, and is known as the formula for a unimolecular reaction. Now, it is sometimes stated that when a process has been found to follow this law, we may conclude that it is a simple chemical reaction. But there are, in many cases, other facts that show that the process as a whole must be much more complex and involve physical as well as chemical changes. Thus, all that we are entitled to deduce from obedience to the unimolecular law is that the slowest of the series of processes, which controls the rate of the whole, *may* be a simple chemical reaction. It does not follow that this is the most significant or most controllable part of the phenomenon.

Equal caution is necessary in regard to conclusions drawn from temperature coefficients. Certain experiments on the rate of the heart-beat gave values directly proportional to the absolute temperature, just as if it were a simple physical process, like the expansion of a gas. Others gave the usual value for

a chemical reaction. Now we know quite well that the contraction of the heart muscle is neither one of these nor the other alone. It seems as if under some conditions the physical factors were in control of the rate; under others, the chemical factors. In any case, we are led to further work.

It will be clear from what has been said that the expression of experimental data in a mathematical formula may have two desirable results. It may serve as a suggestion of a nature of a process, and thus lead to additional experimental tests of the validity of such a hypothesis, or it may for the time being merely serve to control the accuracy of the methods used. If the data fall into no sort of regular form, it may reasonably be suspected that the way in which they were obtained was not free from objection. This point of view is well put in the following quotation from the lectures by Arrhenius on "Immunochemistry" (p. 7). After showing that the process of immunisation can be expressed by a formula, viz.

$$\frac{1}{(\text{concentration})^{n-1}} = \text{const}_1 + \text{const}_2 \cdot t,$$

Arrhenius proceeds: "The application of the formula of Madsen teaches us much more" (that is, than that the quantity of antitoxin in the blood decreases more rapidly in the early stages than later on). "It shows that the phenomenon is a regular one, and we are impelled to seek for a cause for the differences of the values of the constants n and const_2. For instance, the different values of const_2 for the three days in the experiments of Bomstein—are they really different, or do the observed differences depend only on experimental errors? This and other questions suggest themselves after the use of such an equation, and they lead to improvement in the experimental methods, and to very sharp and well-defined ideas of the natural phenomena themselves. With the help of formulæ, which may be empiric or rational, scientific progress will be much more rapid than without them: and as the experimental material increases, the empiric formulæ will probably be converted into rational ones, *i.e.* we shall detect new laws of nature. It is, therefore, very much to be regretted that efforts have been made, especially recently, to reject the use of formulæ in the treatment of questions of serum-therapy. These efforts may be regarded as a last desperate struggle against the stringent conclusions that may be reached by means of the application of mathematical treatment—a struggle that cannot be greatly prolonged."

There appear to be some men of science who are quite satisfied to see their experimental results expressed in the form of a mathematical expression, even when this is merely an empirical one, and ask for nothing more. But most of us want to have some mental picture of what is actually happening, or at least to know the meaning of the factors of our equations in terms of other known phenomena. Thus, although the Barcroft-Hill hæmoglobin formula was at first only the equation to the curve of experimental results, it was not long before attempts were made to find the meaning of the constants—attempts which are still in progress. We see indeed how the formula has led to an immense number of valuable experimental results. The want of satisfaction above alluded to may be felt more especially in relation to conclusions drawn from thermodynamical considerations. While such conclusions are always valid in respect to the direction in which changes take place and the amount of such changes, they are unconcerned with the particular way in which the results are brought about. There is no doubt, on the other hand, as to the value of this mode of treatment, although the mechanism at work is left for future discovery.

Dr. Feldman's book concludes with an account of the most important statistical methods. This will be welcomed. In many domains of biological inquiry this is clearly the only way to arrive at reliable conclusions. At the same time, and without wishing to undervalue the method in its proper sphere, I feel bound to enter a word of protest against the uncritical use of correlation formulæ in experimental work where we have the conditions under control. It must be familiar to all who make such experiments that some of them are obviously of little or no value, while others are so good that their results far outweigh a large number of the bad ones. Indeed, it sometimes happens that one good experiment alone is sufficient to solve the problem. I assume, of course, that we know why this is so, and that we do not call an experiment a bad one merely because it gives results contrary to what we had expected or desired. Claude Bernard warns us "In physiology, more than anywhere else, on account of the complexity of the subjects of experiment, it is easier to make bad experiments than to be certain what are good experiments—that is to say, comparable with one another."

In what criticism I have made in the preceding remarks, it is far from my intention to detract from the value of mathematics in biology, which is shown to be sufficiently evident. To make exaggerated claims or an inappropriate use of the

powerful tool at our disposal seems to me, however, to invite opposition from those who are unsympathetic, and to bring discredit on an extremely valuable help.

I would conclude by strongly recommending all those who wish to follow modern work in the biological field to make themselves familiar with the contents of Dr. Feldman's book.

W. M. BAYLISS.

UNIVERSITY COLLEGE,
LONDON.
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CONTENTS.

BIOMATHEMATICS.

CHAPTER I.

INTRODUCTORY.

Mathematics teaches us, to begin with, the art of computation —of rapid and accurate computation; but Mathematics is far more than an arithmetical art. It is a mode of reasoning, the special kind of Logic by which we measure and compare all manner of magnitudes, all quantitative expressions involving number, size, position, time, all such relations as the more, the equal or the less, and by which we learn to state these relations with the utmost economy of words, of symbols and of thought itself. We do this sometimes by what is called a *formula*, sometimes by means of a diagram—a line, a surface, or a solid. We are applying the methods of mathematics when we wrap up the composition of a chemical substance in a formula, when we express a varying rate or a varying force by means of a curve, or when we define the form of a soap-bubble, a bee's cell or a crystal, and so pave the way to an understanding of the causes which produced, or the *laws* which are said to "govern," these objective phenomena. **Biomathematics** is this art of expression and this science of reasoning as applied to the study of biological facts and problems.

It is the aim of this book to give a concise, but clear and adequate, exposition of those mathematical principles and manipulations a knowledge of which is essential for an intelligent appreciation of the records of modern investigations in the various fields of biological science. Each important mathematical process will, as far as possible, be illustrated by means of examples taken from one or other branch of biological science. This should help the reader to follow the various mathematical steps with greater interest than he would otherwise do. But although the attempt will be made to smooth away difficulties, it would be wrong to pretend that the manner in which the matter is here treated will enable the reader, who is totally new to the subject, to follow what is said with the greatest ease. It is believed, however, that if the reader will set himself

resolutely to the task, and with pen in hand and paper in front of him will study carefully the various processes, step by step, he will be able to overcome the difficulties which are bound to confront the non-mathematical reader every now and again.

Mathematical Accuracy.—The word "accurate" in our definition of "mathematics" requires a little elaboration. Since most of the quantities with which we have to deal in the various branches of biology have been obtained as the result of weighing and measuring, it is clear that the accuracy of the calculation must necessarily depend upon the accuracy with which these data have been obtained. But no measurement can ever be absolutely exact, however skilful and painstaking the observer may have been who made it, and however refined may have been the apparatus with which the observation was made. For example, when we say that the height of a person is 5 ft. $6\frac{3}{4}$ in. we mean that if we place a measuring rod vertically against the individual in question, a horizontal bar from the top of the person's head will meet the measuring rod at the division corresponding to 5 ft. $6\frac{3}{4}$ in. from the ground. The expression 5 ft. $6\frac{3}{4}$ in. implies that we have divided our magnitude into 267 parts each of which is equal to $\frac{1}{4}$ in., and that we claim to measure these to within a unit *more or less*. For, were we to examine the line of contact of the horizontal bar with the measuring rod by means of a magnifying glass, we would see that it does not exactly coincide with the division corresponding to 5 ft. $6\frac{3}{4}$ in. Were we further to translate 5 ft. $6\frac{3}{4}$ in. into its decimal equivalent, 5·5625 ft., we should be still farther from asserting that this was the person's actual height, for to say so without qualification would be to declare that we had measured the precise number of units each one so small that over 55,000 of them were contained in the magnitude in question. The decimal expression would, therefore—without explanation—unjustifiably claim an accuracy about 200 times as great as the first expression implied.

Taking another example: when we say that a cubic millimetre of blood contains 5,000,000 red blood corpuscles, we do not mean that 5,000,000 is the exact number; what we mean is that the number is correct to within, say, 100,000 corpuscles above or below that figure, that is to say to one part in fifty. Indeed, very few measurements, however skilful the observer, are correct to more than 4 figures, or to one part in a thousand, although it is true that with the aid of very delicate apparatus it is possible to obtain a far greater degree of accuracy than this. Thus A. V. Hill has described an apparatus which can

detect a difference of temperature of one hundred millionth of a degree Centigrade (10^{-8}° C.), and Tashiro has studied metabolism in nerves with an apparatus which can detect a current change as minute as one billionth part (10^{-12}) of the initial current of a conductor, while Sir Jagadis Bose's *crescograph*, which he used in the study of growth of plants, is so delicate that it is claimed by its inventor to register 1/100,000 in.—which is the increment in length of the plant per half-second. But even such exceedingly delicate instruments do not give *absolutely exact* measurements. Hence it is clear that in all our calculations the highest accuracy we can and need attain is to a certain number of figures commensurate with the accuracy of the relevant data, and although it is sometimes necessary to calculate results to 6 or 7 figures, an accuracy of 1 in 1000, *i.e.* up to 4 figures, is sufficient for most purposes.

This, however, does not mean that the mathematical processes that we are about to describe are merely approximations. What it does mean is that our mathematical operations, although *with data which are absolutely exact will yield results which are absolutely exact*, need not be carried to a higher degree of accuracy than to a certain number of significant figures, *i.e.* to no more than the number of figures to which the data have originally been ascertained.

Having grasped this essential fact, the reader will understand that if we are confronted with such an arithmetical operation as the multiplication of 2·5968 by 1·7324, and we are told that each of these numbers is known to be absolutely correct only as far as the fourth significant figure (or the third decimal place in this particular instance), whilst the fourth decimal figure is only approximately correct, it would be absurd to record the product to eight decimal places (as the product carried out by the ordinary methods would yield). Indeed, the product obviously can be correct only to 3 decimal places. Hence, one of the first things with which we shall have to deal in these pages is the method of **approximation,** it being understood that the approximation can be carried as near exactness as we like, or are entitled to carry it.

Degree of Accuracy in Calculation.—The statement that a certain calculation has been carried to so many decimal places gives no idea of the accuracy of the result. For instance, when we say that the width of a microscopic object, such as a bacillus, is found to be 0·00004 in., the measurement goes to five decimal places, and yet an error of 1 in the last figure would mean an error of no less than 25 per cent. below or

above the real width. But when we speak of the number of red corpuscles in the human body as being 25,000,000,000,000, an error of a few hundred or even a few thousand million corpuscles is of no consequence. Hence we see that *the degree of accuracy depends not so much on the number of decimal places as on the number of significant figures* to which the result is carried.

Note.—By the term "significant figures" is understood the figures which are definitely known, counting from the left, and omitting all the zeros at the beginning. Thus, 0·00001937 and 193·7 have four significant figures each, while the numbers 0·000019 and 3572·42 have two and six significant figures respectively.

The statement that a measurement is correct to four significant figures means that the first three are absolutely correct and the last one is the nearest figure to the truth. Thus, both 32·174 and 32·166 if given to four significant figures would be 32·17, because 32·166 is nearer to 32·17 than to 32·16, and 32·174 is nearer to 32·17 than to 32·18. Similarly, by common convention, 32·165 would also be given as 32·17 to four significant figures. Hence we see that when we say that 32·17 is correct to four significant figures we mean, strictly speaking, that the number is less than 32·175 but equal to or more than 32·165. In other words, the possible error is not more than 0·005 "in excess or defect," *i.e.* not more than $\pm 0·005$ (read as "plus or minus 0·005"). The *limits of error*, therefore, of 32·17 when described as known to four significant figures are $\pm 0·005$. The actual error may, of course, have any value between these extremes.

CHAPTER II.

LOGARITHMS.

To illustrate the importance of simplifying arithmetical processes we will consider the calculation alluded to on p. 3. The sides of a rectangular object are found to be 2·5968 and 1·7324 units respectively, each measurement being correct to four significant figures, or to three decimal places. What is the area of the rectangle and to how many decimal places is it correct?

Ordinary multiplication of the numbers makes the product equal to 4·49869632, but as the original numbers are correct only to three decimal places it is obvious that the product cannot be correct to all the eight decimal places, and our task is to ascertain to how many places it is correct.

Now, the statement that 2·5968 is correct to four significant figures or three decimal places means that the measurement lies between 2·59675 and 2·59685 (see p. 4). Similarly, 1·7324 stands for some figure between 1·73235 and 1·73245. Hence the product cannot be less than $2{\cdot}59675 \times 1{\cdot}73235$, *i.e.* than 4·4984798625, nor more than $2{\cdot}59685 \times 1{\cdot}73245$, *i.e.* than 4·4989127825. Taking these products to four places of decimals only—since up to the third decimal place the two products agree, we get as follows:—

$$2{\cdot}59675 \times 1{\cdot}73235 = 4{\cdot}498\mathit{5}(= 4{\cdot}4987 - 0{\cdot}0002) = \text{minimum area},$$
$$\mathit{2{\cdot}5968} \times \mathit{1{\cdot}7324} = \mathit{4{\cdot}4987},$$
$$2{\cdot}59685 \times 1{\cdot}73245 = 4{\cdot}498\mathit{9}(= 4{\cdot}4987 + 0{\cdot}0002) = \text{maximum area}.$$

Hence, the area of the rectangle, whose sides 2·5968 and 1·7324 are known to be correct to three decimal places each, is correct to three places only, but the error in the fourth place is not greater than 2. Hence we say that the *limit of error* is $\pm$ 0·0002, and we state that "the product of 2·5968 by 1·7324, when each measurement is known to be correct to three decimal places, is 4·4987 $\pm$ 0·0002."

That the limit of error is $\pm 0{\cdot}0002$ can be shown without actual multiplication by the following method which will probably appeal to those who appreciate mathematical elegance.

The measurement 2·5968 lies between (2·5968 − 0·00005) and (2·5968 + 0·00005). Similarly, 1·7324 lies between (1·7324 − 0·00005) and (1·7324 + 0·00005). Therefore the value of the area of the rectangle must lie between (2·5968 − 0·00005)(1·7324 − 0·00005) and (2·5968 + 0·00005)(1·7324 + 0·00005),

i.e. between $2{\cdot}5968 \times 1{\cdot}7324 - \mathit{0{\cdot}00005(2{\cdot}5968 + 1{\cdot}7324)} + 0{\cdot}0000000025$
and $2{\cdot}5968 \times 1{\cdot}7324 + \mathit{0{\cdot}00005(2{\cdot}5968 + 1{\cdot}7324)} + 0{\cdot}0000000025$.

But these two values differ only in the *sign* of the middle term, viz. $\pm 0{\cdot}00005 (2{\cdot}5968 + 1{\cdot}7324)$, *i.e.* $\pm 0{\cdot}00005 \times 4{\cdot}3292$, or $\pm 0{\cdot}00021646$. Hence, the area is correct to three places only, but the error in the fourth place is not greater than 2, and the *limit of error* is $\pm 0{\cdot}0002$.

It is clear, therefore, that to do the multiplication in the ordinary way and get eight decimal places, when only four places are needed, and only three of which are absolutely correct, is utter waste of time and labour. Hence in operations of this kind one adopts some contracted method which gives the answer to the requisite number of places. The simplest time and labour saving method is by means of **logarithms,** although other methods are described in books on elementary arithmetic.

To understand clearly the meaning and use of logarithms it is necessary to refresh one's memory regarding the meaning and laws of indices.

Indices.—The expression a^m means the product of m factors each of which is equal to a. Thus

$$a^m = a \times a \times a \times a \times a \ . \ . \ . \text{ to } m \text{ factors.}$$

The letter a is called the *base* and the letter m the *index* of the power to which the base a is raised. a^m is read "a to the power m." Thus, 10^4 is read "10 to the power 4" and is equal to $10 \times 10 \times 10 \times 10$ (*i.e.* to 4 factors).

Laws of Indices.—The following laws are universally true, whether a, m and n be positive or negative, integral or fractional:—

1. $a^m \times a^n = a^{m+n}$. *E.g.* $10^2 \times 10^3 = 10^5$ (*i.e.* $100 \times 1{,}000 = 100{,}000$).
2. $a^m/a^n = a^{m-n}$. *E.g.* $10^5/10^3 = 10^2$ (*i.e.* $100{,}000/1{,}000 = 100$).
3. $(a^m)^n = a^{mn}$. *E.g.* $(10^2)^3 = 10^6$, [*i.e.* $(10^2)^3 = 100^3 = 1{,}000{,}000$].
4. $(a^m)^{1/n} = a^{m/n}$. *E.g.* $(1000^2)^{1/3} = 1000^{2/3} = 100$ (see below).

The Meaning of a Fractional Index.—By the first law of indices

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a.$$

$$\text{But} \qquad \sqrt{a} \times \sqrt{a} = a,$$

$$\therefore \quad a^{\frac{1}{2}} = \sqrt{a}.$$

Similarly, $a^{\frac{1}{3}} = \sqrt[3]{a}$; $a^{\frac{1}{4}} = \sqrt[4]{a}$;

and generally, $a^{1/n} = \sqrt[n]{a}$;

i.e. a fractional index of the power of any number means some root of that number. Thus $10^{\frac{1}{2}}$ or $10^{0\cdot 5} = \sqrt{10} = 3\cdot 1623$; $10^{\frac{1}{3}}$ or $10^{0\cdot 3333} = \sqrt[3]{10} = 2\cdot 1544$; $10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}}$ (by the third law) $= \sqrt{3\cdot 1623} = 1\cdot 7783$; and so on.

The Meaning and Value of a^0.—By successively halving the index of the power of 10, *i.e.* by extracting its 4th, 8th, 16th, etc., roots, we obtain as follows:—

$$10^{\frac{1}{4}} \text{ or } 10^{0\cdot 25} = \sqrt{3\cdot 1623} = 1\cdot 7783$$
$$10^{\frac{1}{8}} \text{ or } 10^{0\cdot 125} = \sqrt{1\cdot 7783} = 1\cdot 3335$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$10^{1/128} \text{ or } 10^{0\cdot 0078} = 1\cdot 0184$$
$$10^{1/256} \text{ or } 10^{0\cdot 0039} = 1\cdot 0093$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$10^{1/2048} \text{ or } 10^{0\cdot 0005} = 1\cdot 001$$

and so on.

In other words, as the fractional index $1/n$ becomes less and less, $10^{1/n}$ becomes nearer and nearer 1; but it does not become absolutely equal to 1 until n becomes infinitely large and $1/n$ therefore becomes zero. Hence, *$10^0 = 1$.*

Similarly, if instead of 10 we take any other number, we get the same result, so that *$a^0 = 1$, whatever number "a" stands for.*

That $a^0 = 1$ can also be proved as follows:—

$$a^m/a^n = a^{m-n} \text{ (second law of indices).}$$
$$\therefore \quad a^m/a^m = a^{m-m} = a^0.$$
$$\text{But} \quad a^m/a^m = 1.$$
$$\therefore \quad a^0 = 1.$$

The meaning of $a^{m/n}$ is that a is to be raised to the mth power and is then to have the nth root extracted. *E.g.*

$$10^{3/4} \text{ or } 10^{0\cdot 75} = \sqrt[4]{10^3} = \sqrt[4]{1000} = 5\cdot 6234;$$
$$10^{4/3} \text{ or } 10^{1\cdot 3333} = \sqrt[3]{10^4} = \sqrt[3]{10{,}000} = 21\cdot 54.$$

The Meaning of a Negative Index.—Since $a^0 = 1$, therefore $1/a^m = a^0/a^m = a^{0-m} = a^{-m}$. Therefore $a^{-m} = 1/a^m$. Thus $10^{-1} = 1/10$; $10^{-2} = 1/100$; etc.

Logarithms.—From the foregoing it will have been gathered that all numbers can be represented as being some power of 10. Thus $1 = 10^0$; $100 = 10^2$; $1/10$ or $0{\cdot}1 = 10^{-1}$; $1/1000$ or $0{\cdot}001 = 10^{-3}$; $1{\cdot}0184 = 10^{0\cdot0078}$ (p. 7), whence also it follows that, say, $10{,}184 = 10^{0\cdot0078} \times 10^4 = 10^{4\cdot0078}$; and so on. Now, *whenever we express a number "a" in the form $a = 10^m$, we say that "m" is the logarithm of "a" to the base 10, and it is written $m = log_{10}\, a$.* Thus 2 is the logarithm of 100, ($2 = \log 100$), because $100 = 10^2$; 0·0078 is the logarithm of 1·0184 because $1{\cdot}0184 = 10^{0\cdot0078}$.

We shall see presently (p. 81) that it is possible to calculate the logarithm of any given number to as many places of decimals as desired. A table of four-figure logarithms, sufficient to answer most of the purposes of this book, is given at the end of the volume. The use of such tables reduces complicated and laborious arithmetical manipulations such as multiplications, divisions, extraction of roots, etc., to *simple* additions and subtractions.

Use of Logarithms in Arithmetical Computation.—As an example, we shall take the multiplication of 2·5968 by 1·7324 again. Five-figure logarithm tables give the logarithms of these numbers as 0·41444 and 0·23865 respectively. This means that $2{\cdot}5968 \times 1{\cdot}7324 = 10^{0\cdot41444} \times 10^{0\cdot23865} = 10^{0\cdot65309}$ (by the first law of indices). Looking up the tables again we find that 0·65309 is the logarithm of 4·4987. Hence we see that *simple addition of the logarithms of two (or more) numbers gives the logarithm of the product of the numbers, from which the value of the product itself is at once ascertained.* This is expressed symbolically as follows:—

$$\log abcd \ldots = \log a + \log b + \log c + \log d + \ldots \quad (\textit{Rule I.})$$

Similarly, *the logarithm of the quotient of two numbers (or any fraction) is obtained by subtracting the logarithm of the denominator from that of the numerator*; or expressed symbolically:—

$$\log a/b = \log a - \log b. \quad (\textit{Rule II.})$$

Thus $2{\cdot}5968/1{\cdot}7324 = 10^{0\cdot41444}/10^{0\cdot23865} = 10^{0\cdot17579}$ (second law of indices) which equals 1·4990.

Further, *n times the logarithm of a given number gives the logarithm of the nth power of that number*; or symbolically:—

$$\log a^n = n \log a. \quad (\textit{Rule III.})$$

Thus $1{\cdot}7324^3 = (10^{0\cdot23865})^3 = 10^{0\cdot71595} = 5{\cdot}1994$ (see third law of indices).

Finally, *the logarithm of the nth root of a given number is equal to 1/n of the logarithm of that number*; or symbolically:—

$$\log a^{1/n} \quad \text{or} \quad \log \sqrt[n]{a} = \frac{1}{n} \log a.$$

Thus $\sqrt[5]{2{\cdot}5968} = 10^{0\cdot41444/5} = 10^{0\cdot08289} = 1{\cdot}2103.$

Characteristic and Mantissa.—The integral part of any logarithm is called the **characteristic**; the decimal portion is called the **mantissa**. Thus, if log 15·2 is 1·1818, then 1 is the characteristic and 0·1818 is the mantissa.

Since log 1 = 0 and log 10 = 1, it is clear that all numbers lying between 1 and 10 must have logarithms lying between 0 and 1, *i.e.* have a mantissa but no characteristic (or have a zero characteristic). *E.g.* log 2 = 0·3010 and log 9 = 0·9542. Similarly all numbers between 10 and 100 must have a logarithm in which the characteristic is 1, but a mantissa lying between 0 and 1. Thus log 11 = 1·0414, and log 99 = 1·9956. Similarly the characteristic of the logarithm of any number between 100 and 1000 is 2, although the mantissa is different in each case; and so on. Moreover, consider such a number as 9·458. Its logarithm is 0·9758, which means that $10^{0\cdot9758} = 9{\cdot}458$. Then, by the laws of indices, we have

$$\begin{aligned} 10^{0\cdot9758} \times 10 \quad &\text{or} \quad 10^{1\cdot9758} = 94{\cdot}58 \\ 10^{0\cdot9758} \times 10^2 \quad &\text{or} \quad 10^{2\cdot9758} = 945{\cdot}8 \\ 10^{0\cdot9758} \times 10^3 \quad &\text{or} \quad 10^{3\cdot9758} = 9458 \end{aligned}$$

and so on.

Again,

$$\begin{aligned} 10^{0\cdot9758}/10 \quad &\text{or} \quad 10^{0\cdot9758-1} = 0{\cdot}9458 \\ 10^{0\cdot9758}/10^2 \quad &\text{or} \quad 10^{0\cdot9758-2} = 0{\cdot}09458 \end{aligned}$$

Hence we see that *all numbers which contain the same significant figures in the same order, and which differ only in the position of the decimal point, have the same mantissa in their logarithms, but different characteristics.* Hence, a knowledge of the logarithm of any number gives us at once the logarithm of the same number multiplied or divided by any power of 10. Another fact that we learn from the foregoing is that it is most convenient to write the mantissa in the form of a *positive* decimal. Thus we regard log 0·9458 as 0·9758 − 1, and not as −0·0242, because in the form 0·9758 − 1 the mantissa is the same as in log 9458 or in log 9·458, etc. But instead of writing the logarithm of a

decimal number in the form say $0·9758 - 1$, one writes it as $\bar{1}·9758$. Similarly, $\log 0·09458 = \bar{2}·9758$, the negative sign being placed *over* the characteristic instead of in front of it, showing that it is only the characteristic and not the whole logarithm that is negative. Thus while $-2·9758$ stands for $-2-0·9758$, $\bar{2}·9758$ stands for $-2+0·9758$.

Note.—$\bar{1}·9758$ is read "bar 1, point 9758"; $\bar{2}·9758$ is read "bar 2, point 9758"; and so on.

Use of Logarithm Tables.—Supposing we wish to find the logarithm of 3158, and of 3·158 and of 0·003158. Look down the first or left-hand column of the table until you come to the figures 31. Then, pass horizontally to the right until the 7th column is reached, at the head of which stands the figure 5. It is thus found that the mantissa of any number whose first three significant figures are 315 is 4983. Now search among the last nine columns for that at the head of which stands the figure 8, and in the horizontal line corresponding to the mantissa just found we find the number 11. This must be added to the last figures in the mantissa, making the latter equal to 4994, which means that $\log 3·158 = 0·4994$. Hence the logarithms we seek are $\log 3158 = 3·4994$, $\log 3·158 = 0·4994$, $\log 0·003158 = \bar{3}·4994$. In all these logarithms the fourth decimal figure is correct to the nearest unit.

The Principle of Proportional Parts.—The four-figure logarithm table gives logarithms to four decimal places of numbers of up to four digits, *i.e.* 1 to 9,999. It is possible, however, by its means to calculate the logarithm of a number consisting of five digits by the principle of proportional parts, which is: *For numbers differing by small quantities, say by less than 1 in 1,000, the differences of the logarithms are proportional to the differences of the numbers.* Thus, suppose we wish to find the logarithm of 3355·7. From the table we find $\log 3355 = 3·5256$, and $\log 3356 = 3·5258$. Therefore a difference of 1 in the numbers corresponds to a difference of 0·0002 in the logarithms, and hence a difference of 0·7 in the numbers must correspond to a difference of $0·7 \times 0·0002 = 0·00014$ or approximately 0·0001 in the logarithms. Therefore $\log 3355·7 = 3·5256 + 0·0001 = 3·5257$.

Note.—When four-figure logarithm tables are used in computation, the results cannot be relied on to more than *three* decimal places. Thus the product of five quantities involves the addition of five logarithms, each of which is only correct to the nearest unit in the fourth place. The sum may therefore be incorrect to the extent of 2 or 3 in the fourth place of

decimals, which may be equivalent to an error of 7 or 8 in the fourth figure in the answer. For more accurate work therefore more accurate logarithm tables must be used.

EXAMPLES.

(1) If $\log 317 = 2{\cdot}5011$ and $\log 318 = 2{\cdot}5024$, what is $\log 317{\cdot}2$?

Difference of 1 in the numbers gives a difference of 0·0013 in the logarithms.

$\therefore$ Difference of 0·2 in the numbers gives a difference of $0{\cdot}0013 \times 0{\cdot}2 = 0{\cdot}0003$ in the logarithms.

$\therefore$ $\log 317{\cdot}2 = 2{\cdot}5011 + 0{\cdot}0003 = 2{\cdot}5014$.

(2) The duration, t (in decimals of a second), of the presphygmic period of a ventricular systole is given by $t = 2\mathrm{R}\left[\left(2{\cdot}302585p\mathrm{R}\log\frac{p\mathrm{R}}{p\mathrm{R}-1}\right)-1\right]$, where p = rate of change of intraventricular pressure per second, and R is a constant depending upon the size of the valvular leak (see p. 337). If $p = 20$ metres water per second and $\mathrm{R} = 2{\cdot}890134$ (for a leak 1 square centimetre in area), what is the value of t?

Substituting these values we get

$$t = 5{\cdot}780268\left[\left(2{\cdot}302585 \times 57{\cdot}80268 \log\frac{57{\cdot}80268}{56{\cdot}80268}\right)-1\right].$$

Now, $\log\frac{57{\cdot}80268}{56{\cdot}80268} = \log 57{\cdot}80268 - \log 56{\cdot}80268 = 1{\cdot}7619480 - 1{\cdot}7543688$

$= 0{\cdot}0075792$ (using seven-figure logarithms).

$\therefore$ Expression in round brackets $= 2{\cdot}302585 \times 57{\cdot}80268 \times 0{\cdot}0075792 =$ say, x.

$\therefore$ $\log x = \log 2{\cdot}302585 + \log 57{\cdot}80268 + \log 0{\cdot}0075792$
$= 0{\cdot}3622157 + 1{\cdot}7619480 + \bar{3}{\cdot}8796234 = 0{\cdot}0037871$.

$\therefore$ $x = 1{\cdot}00874$.

$\therefore$ Expression in square brackets $= 1{\cdot}00874 - 1 = 0{\cdot}00874$.

$\therefore$ $t = 5{\cdot}780268 \times 0{\cdot}00874$.

$\therefore$ $\log t = \log 5{\cdot}780268 + \log 0{\cdot}00874 = 0{\cdot}7619480 + \bar{3}{\cdot}9415114 = \bar{2}{\cdot}7034594$.

$\therefore$ $t = 0{\cdot}05052$ second.

As the isometric period in a normal heart is 0·05 second, therefore its duration in the leaking heart (in which the area of the leak is 1 square centimetre) is about 1 per cent. longer (see p. 337).

(3) The formula $\frac{\mathrm{V}_{t+n}}{\mathrm{V}_t} = x^n$ represents the influence of temperature upon the velocity of a chemical reaction, V_t being the velocity at temperature $t°$ C., V_{t+n} that at $(t+n)°$ C., and x the increase in velocity per 1° C. It took a developing tadpole ovum $3\frac{1}{3}$ times as long to reach a certain stage of development at 10° as it took at 20° C. What is the increase of the rate of growth (x) per 1° C. between the two temperatures?

From the formula we have $\mathrm{V}_{20}/\mathrm{V}_{10} = 3{\cdot}333 = x^{10}$.

$\therefore$ $10 \log x = \log 3{\cdot}333 = 0{\cdot}5228$.

$\therefore$ $\log x = 0{\cdot}0523$.

$\therefore$ $x = 1{\cdot}13$.

(4) Robertson found that the decay of memory traces for meaningless syllables committed to memory is expressed by the equation

$$\mathrm{N} - n = 1{\cdot}733t^{0{\cdot}056}$$

where N = number of syllables originally committed to memory and n is the number of syllables remembered after t hours. Find after what time all of 13 syllables committed to memory will be completely forgotten.

When all syllables are forgotten $n = 0$.

$$\therefore \quad 13 = 1{\cdot}733 t^{0{\cdot}056}.$$
$$\therefore \quad \log 13 = \log 1{\cdot}733 + 0{\cdot}056 \log t,$$
$$i.e. \quad 1{\cdot}1139 = 0{\cdot}2387 + 0{\cdot}056 \log t.$$
$$\therefore \quad \log t = \frac{1{\cdot}1139 - 0{\cdot}2387}{0{\cdot}056} = 15{\cdot}6286.$$

But $0{\cdot}6286$ is $\log 4{\cdot}252$.

$$\therefore \quad 15{\cdot}6286 = \log 4{\cdot}252 \times 10^{15}.$$
$$\therefore \quad t = 4252 \times 10^{12} \text{ hours} = 485 \times 10^{9} \text{ years}.$$

Hence, memory traces are never completely wiped out, and although they may be suppressed during the waking state, they may reveal themselves during a state of hypnosis when cerebral inhibition is removed.

(5) Dreyer found the following relationship between W (weight in grammes) and λ (sitting height in centimetres) of a person: $W = (0{\cdot}3803\lambda)^{\frac{1}{0{\cdot}319}}$. Find the sitting height of a person weighing 89·78 kilograms.

$$\log 89780 = \frac{\log 0{\cdot}3803 + \log \lambda}{0{\cdot}319},$$

i.e.

$$4{\cdot}9532 = \frac{\bar{1}{\cdot}5801 + \log \lambda}{0{\cdot}319}.$$
$$\therefore \quad \log \lambda = 2.$$
$$\therefore \quad \lambda = 100 \text{ cm}.$$

(6) The following equation (Du Bois) represents the relationship between the surface area of the body in square centimetres (S) and its height in centimetres (H) and weight in kilograms (W): $S = 71{\cdot}84\, W^{0{\cdot}425}\, H^{0{\cdot}725}$. Find the surface area when W = 60 kg. and H = 150 cm.

$$\log S = \log 71{\cdot}84 + 0{\cdot}425 \log W + 0{\cdot}725 \log H$$
$$= 1{\cdot}8563 + 0{\cdot}425 \times 1{\cdot}7782 + 0{\cdot}725 \times 2{\cdot}1761 = 4{\cdot}1897.$$
$$\therefore \quad S = 15{,}470 \text{ sq. cm.} = 1{\cdot}55 \text{ square metres}.$$

(7) The number of bread eaters in the world has been increasing 12 per cent. in a decade, and in 1898 it was about 516 millions. The utmost area of land available for the growth of wheat was estimated at 263 million acres capable of producing 12·7 bushels per acre per year. If each bread eater needs $4\frac{1}{2}$ bushels of wheat per year, in what decade will an average harvest be insufficient to meet the demand?

Maximum annual yield of wheat $= 263 \times 12{\cdot}7 = 3340$ million bushels, which is enough to feed $3340/4\frac{1}{2} = 742{\cdot}2$ million bread eaters.

The 516 million bread eaters will increase to $516(1{\cdot}12)$ million in one decade, $516(1{\cdot}12)(1{\cdot}12) = 516(1{\cdot}12)^2$ in two decades, and to $516(1{\cdot}12)^x$ in x decades. If therefore we equate $516(1{\cdot}12)^x$ to $742{\cdot}2$, the value of x thus found gives the requisite decade.

Thus $$\log 516 + x \log 1{\cdot}12 = \log 742{\cdot}2.$$
$$\therefore \quad 2{\cdot}7126 + 0{\cdot}0492x = 2{\cdot}8705,$$

whence $x = 3{\cdot}21$ decades, *i.e.* the harvest will be insufficient in the 33rd year after 1898, or in the year 1930.

As there is still enough bread to feed everybody at the present day this example shows the futility of making estimates of population and amount of available food too many years in advance from data available at the moment.

(8) The population of a certain country doubles itself in 100 years. Find the rate of growth, r, assuming it to be constant. If the population is a million at the beginning of the century, what will it be in 20, 50 and 80 years respectively?

Let r = percentage rate of increase per year.

Then, at the end of one year the million population becomes $1{,}000{,}000\left(1+\frac{r}{100}\right)$. This becomes $1{,}000{,}000\left(1+\frac{r}{100}\right)^{100}$ in 100 years (see Example (7)).

$$\therefore \quad \left(1+\frac{r}{100}\right)^{100} = 2.$$

$$\therefore \quad 100 \log\left(1+\frac{r}{100}\right) = \log 2 = 0{\cdot}3010.$$

$$\therefore \quad \log\left(1+\frac{r}{100}\right) = 0{\cdot}0030 = \log 1{\cdot}007.$$

$$\therefore \quad r/100 = 0{\cdot}007,$$

whence $r = 0{\cdot}7$ per cent. or 7 per 1000 per year.

In 20 years the million population will become $1{,}000{,}000\,(1{\cdot}007)^{20} = x$, say.

$\therefore\quad \log x = 20 \log 1{\cdot}007 + \log 1{,}000{,}000 = 6{\cdot}058 = \log 1{,}143{,}000.$

$\therefore\quad$ Population in 20 years = 1,143,000 approximately.

Similarly, it will be found that in 50 years time the population will be 1,396,000, and in 80 years 1,706,000.

Logarithms to a Base other than 10.—In the same way as we can express any number as some power of 10, it is possible to express any number also as some power of any other base or fundamental number, such as of 2, 3, 4, etc. Thus $4 = 2^2$; $1{\cdot}4142$ which is $\sqrt{2} = 2^{0{\cdot}5}$; etc. Hence we can say that 2 is the logarithm of 4 to the base 2, and 0·5 is the logarithm of 1·4142 to the base 2. Indeed, whenever we have an expression like $a^m = b$, we can say that m is the logarithm of b to the base a. The symbolical way of writing this is $m = \log_a b$. Hence strictly, to express that m is the logarithm of b to the base 10, we ought to write $m = \log_{10} b$. But as the ordinary logarithms used in computation are those to the base 10, we simply write $m = \log b$, the base 10 being understood.

Napierian Logarithms.—While for ordinary computation purposes we use logarithms to the base 10—called Common Logarithms—it is convenient as we shall see later (p. 82) in mathematical analysis to use logarithms in which the base is the incommensurable number 2·71828 . . ., generally denoted

by the letter e. This system of logarithms is called the Napierian system, since John Napier, the inventor of logarithms, calculated his logarithms to this base (in 1614). Henry Briggs, in 1617, introduced the common logarithms to the base 10, which are, therefore, sometimes called Briggsian logarithms.

It is obvious that since e is smaller than 10, the index of the power to which e has to be raised in order to produce a given number a must be higher than that to which 10 has to be raised to produce the same number. In other words, the logarithm of a to the base e, ($\log_e a$), must be greater than the logarithm of a to the base 10 or $\log_{10} a$.

Conversion of Logarithms from One System to the Other.—Logarithm tables give log 2·71828 . . . or log e (*i.e.* $\log_{10} e$) as 0·4343, which means that $e = 10^{0\cdot4343}$.

$$\therefore \quad e^m = 10^{0\cdot4343m} = a \text{ (say)}.$$

$$\therefore \quad m = \log_e a, \quad \text{and} \quad 0\cdot4343m = \log_{10} a \text{ (by definition of a logarithm).}$$

Hence $$\log_{10} a = 0\cdot4343 \log_e a,$$

and conversely, $$\log_e a = \frac{1}{0\cdot4343} \log_{10} a$$
$$= 2\cdot3026 \log_{10} a.$$

The figure 0·4343, which converts a Napierian to a common logarithm, is called the *Modulus* of the common logarithm.

EXERCISES.

(1) If the number of persons born in any year is 1/45th of the whole population at the beginning of the year, and the number of those who die 1/60th of it, in how many years will the population be doubled, given log 181 = 2·2577 and log 180 = 2·2553? [*Answer*, 125 years.]

(2) The volume of a cylinder is given by the formula $v = \pi r^2 h$ (where r = radius and h = height). Find v if $r = 0\cdot5$, $h = 12\cdot76$ and $\pi = 3\cdot142$. [*Answer*, 10.]

(3) Meeh's formula for the surface area of the human body is $S = K\sqrt[3]{W^2}$, where S = surface in square decimetres, W = weight in kilograms and K is a constant which for children is 10·3. It has been found by Benedict and Talbot that if l is the length of an infant (cm.), then the amount of heat produced by it in 24 hours is $0\cdot1265\, lK\sqrt[3]{W^2}$. How much heat should an infant produce whose weight is 3·63 kilograms and measuring 52 cm. in length?

[*Answer*, $\log 0\cdot1265 + \log 52 + \frac{2}{3}\log 3\cdot63 + \log 10\cdot3 = 2\cdot2041 = \log 160$. Therefore amount of heat = 160 calories.]

(4) The following are the figures for the rate of growth of pea rootlets:—

At 10° C.	0·41 mm. per hour.
,, 14° C.	0·61 ,, ,,
,, 20° C.	1·01 ,, ,,
,, 24° C.	1·43 ,, ,,

Find the increase in the rate of growth (x) for the temperature intervals $10-20°$, and $14-24°$.

[*Answer*, $V_{20}/V_{10} = 1{\cdot}01/0{\cdot}41 = x_1^{10}$, $\therefore \log x_1 = 1/10 (\log 1{\cdot}01 - \log 0{\cdot}41) = 0{\cdot}03915$, $\therefore x_1 = 1{\cdot}094$. Similarly, $x_2 = 1{\cdot}089$.]

(5) What is the temperature coefficient, *i.e.* $(V_{t+10})/V_t$, see p. 11, for the inversion of cane sugar given $V_{25} = 0{\cdot}765$, and $V_{55} = 35{\cdot}5$.

[*Answer*, $\log x^{10} = \frac{1}{3}(\log 35{\cdot}5 - \log 0{\cdot}765) = 0{\cdot}5555$. $\therefore x^{10} = 3{\cdot}6$.]

(6) The reaction of a solution, *i.e.* the degree of its acidity or alkalinity, is quantitatively expressed by its pH value, which stands for the negative logarithm of cH, or hydrogen-ion concentration. $p\text{H} = 7$ (which is the same as $c\text{H} = 10^{-7}$) means neutrality; $p\text{H} > 7$ means alkalinity; $p\text{H} < 7$ means acidity. Blood has a pH value of 7·45. What is its hydrogen-ion concentration in grammes per litre?

The negative logarithm of 7·45, *i.e.* the logarithm of $-7{\cdot}45$, is the same as that of $\bar{8}{\cdot}55$.

But 0·55 is the logarithm of 3·54.

$\therefore$ $\bar{8}{\cdot}55$ is the logarithm of $3{\cdot}54 \times 10^{-8}$, *i.e.* blood contains $3{\cdot}54 \times 10^{-8}$ grammes of hydrogen-ion per litre.

(7) Two solutions have pH values of 6·4 and 6·7 respectively. What are their relative acidities?

The H-ion concentration corresponding to $p\text{H}\ 6{\cdot}4 = 10^{-6\cdot4}$.

,, ,, ,, ,, $p\text{H}\ 6{\cdot}7 = 10^{-6\cdot7}$.

$\therefore$ The relative H-ion concentrations are $10^{-6\cdot4} : 10^{-6\cdot7} = 10^{0\cdot3} : 1$.

But 0·3 is the logarithm of about 2.

$\therefore$ $10^{0\cdot3} : 1 = 2 : 1$.

$\therefore$ Although the difference in the pH values is comparatively slight, the solution whose pH value is 6·4 is about twice as acid as the one whose pH value is 6·7.

(8) Knowing $\log 2 = 0{\cdot}3010$, find the value of $\log 5$.

[*Answer*, $\log 5 = \log 10/2 = \log 10 - \log 2 = 1 - 0{\cdot}3010 = 0{\cdot}6990$.]

CHAPTER III.

A FEW POINTS IN ALGEBRA.

We will assume that the reader is familiar with the solution of simple algebraical equations, such as $x+3 = 6$, giving $x = 3$; and with the addition and subtraction of fractions such as $\frac{1}{x+1}+\frac{1}{x-1}$, where the fractions have to be reduced to a common denominator $(x+1)(x-1)$ and added, thus:

$$\frac{1}{x+1}+\frac{1}{x-1} = \frac{(x-1)}{(x+1)(x-1)}+\frac{(x+1)}{(x+1)(x-1)}$$
$$= \frac{2x}{(x+1)(x-1)} = \frac{2x}{(x^2-1)}$$

Let us now introduce the reader to two simple but important points, viz.:

(1) The distinction between an *equation* and an *identity.*

(2) The meaning of the term *partial fractions.*

(1) **Equation and Identity.**—Whilst an equation like $x+3 = 6$ or $x^2+4x+4 = 9$ is a statement to the effect that the expressions on either side of the equality sign $(=)$ are equal to each other for certain *particular* values of the unknown quantity (x) used in the expressions—these particular values being called the **roots** of the equations—an *identity* is a statement that two algebraical expressions are equal for *all* values of the unknown quantity.

Thus $x+3 = 6$ expresses the fact that if 3 is substituted for x, the expressions on either side of the equality sign become alike, *i.e.* $3+3 = 6$.

Similarly $x^2+4x+4 = 9$ is true only when $x = +1$ or -5; thus if $x = 1$, we have

$$1+4+4 = 9,$$

and if $x = -5$, we have

$$25-20+4 = 9.$$

But $x+3 = 6$ is not true if we put $x = 1$, or 2, or any other

value except 3; and $x^2+4x+4 = 9$ is not true if we give x any value other than $+1$ or -5.

If, however, we take such an expression as

$$x+3 = 2\cdot\frac{x}{2}+3$$

or

$$x+3 = \tfrac{1}{3}x+\tfrac{1}{3}x+\tfrac{1}{3}x+3$$

or

$$(x-1)(x+1) = x^2-1,$$

such statements are *identities*, because whatever value one gives x in any of these expressions the result is *always* true.

The following identities, which the student can verify by actual multiplication, must be committed to memory, and should be familiar each way:—

$$(a+b)^2 = a^2+2ab+b^2 \quad . \quad . \quad . \quad . \quad (1)$$
$$(a-b)^2 = a^2-2ab+b^2 \quad . \quad . \quad . \quad . \quad (2)$$
$$a^2-b^2 = (a+b)(a-b) \quad . \quad . \quad . \quad . \quad (3)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2) \quad . \quad . \quad . \quad (4)$$
$$a^3+b^3 = (a+b)(a^2-ab+b^2) \quad . \quad . \quad . \quad (5)$$

Another important identity with which we shall deal later is the *binomial theorem* (see p. 66), viz.:

$$(a+b)^n = a^n+na^{n-1}b+\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3+\ .\ .\ . + \frac{n(n-1)}{1\,.\,2}a^2b^{n-2}+nab^{n-1}+b^n.$$

Whatever the values of a, b and n, this result is always true.

These identities will not only crop up continually in the course of our manipulations, but they are sometimes of considerable use for the purpose of simplifying arithmetical operations.

E.g., (1) $22{\cdot}9^2-22{\cdot}1^2 = (22{\cdot}9+22{\cdot}1)(22{\cdot}9-22{\cdot}1)$ by identity (3)
$= 45{\cdot}0\times0{\cdot}8 = 36{\cdot}00.$

(2) Find the square of 1·0013 correct to 5 significant figures.

$(1{\cdot}0013)^2 = (1+0{\cdot}0013)^2 = 1+2\times0{\cdot}0013+(0{\cdot}0013)^2$ by identity (1)
$= 1{\cdot}0026.$

$(0{\cdot}0013)^2$ being equal to 0·00000169 does not affect the 5th significant figure, and may therefore be ignored.

(3) Find the value of $1{\cdot}0013\times0{\cdot}9987$. This can easily be put into the form $(a+b)(a-b) = a^2-b^2$.

Thus $(1+0{\cdot}0013)(1-0{\cdot}0013) = 1^2-0{\cdot}00000169$
$= 0{\cdot}99999831.$

Note.—The student's attention is particularly directed to identities (1) and (2) which are of fundamental importance in the solution of quadratic equations, *i.e.* equations containing as a term the square (but no higher power) of the unknown, *e.g.* $x^2+6x = 7$, or, which is the same thing, $x^2+6x-7 = 0$. It will be observed that if an expression of the second degree is a complete square, then the coefficient of the third term is the square of half the coefficient of the middle term provided the coefficient of the first term is unity. Thus in the given identities, the coefficient of a^2 is 1, and, therefore, the coefficient of b^2 is $\left(\frac{\pm 2}{2}\right)^2 = 1$.

What we have said about identities introduces us to another important point, viz. the factorisation of a complicated algebraical expression.

Law of Factors.—If an expression like

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ .\ .\ .\ +Kx+m$$

contains $(x-a)$ as a factor, then when, in the given expression, x is put $= a$, the expression must become $= 0$.

This is almost obvious. For if $(x-a)$ is a factor of the expression and Q is the other factor, we must have

$$Ax^n+Bx^{n-1}+\ .\ .\ .\ +Kx+m = Q(x-a).$$

This being an identity is true for all values of x.

$\therefore$ It is true also when $x = a$.

Put $x = a$, and we get

$$Aa^n+Ba^{n-1}+\ .\ .\ .\ +Ka+m = Q\times 0 = 0.$$

Hence if $Ax^n+Bx^{n-1}+\ .\ .\ .\ +Kx+m$ vanishes when $x = a$, then it contains $x - a$ as a factor.

EXAMPLES.

(1) Find the factors of $x^3-3x^2-10x+24$.

Put $x = 1$ and the expression becomes

$$1-3-10+24 = 12.$$

As the expression does not vanish, $x-1$ is *not* a factor.

Put $x = 2$ and the expression becomes $8-12-20+24 = 0$,

$\therefore$ $x-2$ is a factor.

Divide $x^3-3x^2-10x+24$ by $x-2$ and the quotient is x^2-x-12, the factors of which are $(x+3)$ and $(x-4)$.

Hence $x^3-3x^2-10x+24 = (x-2)(x+3)(x-4)$.

Or we might have tested for a factor $x+3$ by putting $x = -3$, when the expression becomes

$$-27-27+30+24 = 0.$$

$\therefore$ $x+3$ is a factor.

Similarly by making $x = 4$, the expression vanishes.

$\therefore$ $x-4$ is a factor.

(2) Find the square root of $x^4+4x^3+10x^2+12x+9$.

Let $\sqrt{x^4+4x^3+10x^2+12x+9} = (x^2+ax+3)$.

$\therefore \quad x^4+4x^3+10x^2+12x+9 = (x^2+ax+3)^2$.

$= x^4+2ax^3+(6+a^2)x^2+6ax+9$.

This being an identity the coefficients of like powers of x must be equal to one another.

$\therefore \quad 4x^3 = 2ax^3$, whence $a = 2$.

$\therefore \quad$ the required root $= x^2+2x+3$.

(3) Find the factors of $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

By putting $b = c$, the expression becomes

$$0+b^2(b-a)+b^2(a-b) = 0.$$

$\therefore \quad (b-c)$ is a factor.

Similarly $(c-a)$ and $(a-b)$ are factors.

$\therefore \quad a^2(b-c)+b^2(c-a)+c^2(a-b) = \mathrm{K}(a-b)(b-c)(c-a)$ . . . (1)

where K is the remaining factor.

Now (1) being an identity is true for all values of a, b and c. Put $a = 0$, then we get

$$0+b^2c-c^2b = \mathrm{K}(-b)(b-c)c$$

or $$bc(b-c) = -\mathrm{K}bc(b-c).$$

$$\therefore \quad \mathrm{K} = -1.$$

$\therefore \quad$ Required factors are $-(b-c)(c-a)(a-b)$.

EQUATIONS AND THEIR SOLUTION.

The *degree* of an equation is determined by the index of the highest power of the unknown quantity in the equation.

Thus $ax+b = 0$ is an equation of the first degree (or *simple*).

$ax^2+bx+c = 0$ is an equation of the second degree (or *quadratic*).

$ax^3+bx^2+cx+d = 0$ is an equation of the third degree (or *cubic*).

. . . .

$ax^n+bx^{n-1}+cx^{n-2}+\ldots+lx+m = 0$ is an equation of the nth degree.

Simple Equations.—It is assumed that the reader is familiar with the method of solving a simple equation with one unknown, such as $ax+b = 0$, when $x = -\dfrac{b}{a}$.

Quadratic Equations.—The solution of quadratic equations is very important. There are **three methods** of dealing with such equations, viz.:

(*a*) **Factorisation.**—*E.g.* if $2x^2-5x-3 = 0$, then by factorisation $(2x+1)(x-3) = 0$.

$\therefore \quad$ either $2x+1 = 0$, giving $x = -\frac{1}{2}$,

or $x-3 = 0$, giving $x = 3$.

(*b*) **Application of the Identity** $(x+A)^2 = x^2+2Ax+A^2$ (p. 17). If the factors cannot be easily detected, then we proceed to convert the equation into the form $(x+A)^2$ as follows:—

Let equation be $$ax^2+bx+c = 0.$$

$$\therefore \quad x^2+\frac{b}{a}x+\frac{c}{a} = 0.$$

The expression is divided throughout by a in order to make the coefficient of x^2 unity (see Note on p. 18).

Now transfer the last term to the other side, thus:

$$x^2+\frac{b}{a}x = -\frac{c}{a}.$$

If now we add to each side the square of half the coefficient of x, *i.e.* $\left(\frac{b}{2a}\right)^2$ or $\frac{b^2}{4a^2}$, then the left-hand side will become a perfect square (see Note, p. 18).

Thus $$x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = -\frac{c}{a}+\frac{b^2}{4a^2} = \frac{b^2-4ac}{4a^2}$$

or $$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}.$$

$$\therefore \quad x+\frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a}.$$

$$\therefore \quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Thus in the example given

$$2x^2-5x-3 = 0$$

$$a=2,\ b=-5,\ c=-3.$$

$$\therefore \quad x = \frac{5\pm\sqrt{25+24}}{4} = \frac{5+7}{4} \text{ or } \frac{5-7}{4}$$

$$= 3 \text{ or } -\tfrac{1}{2}.$$

The expression $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for the roots of a quadratic equation leads to the following conclusions:—

(1) If $b^2 = 4ac$, then the expression under the square root sign vanishes, and therefore $x = \frac{-b\pm 0}{2a}$, *i.e.* the two roots are equal.

E.g. in the equation $4x^2+12x+9 = 0$,

$$x = \frac{-12 \pm \sqrt{144-36\times 4}}{8} = -\frac{3}{2},$$

i.e. the two roots are each equal to $-\frac{3}{2}$.

(2) If $b^2 > 4ac$, then the expression under the square root sign is positive, and therefore the equation has two *real* roots.

E.g. in the equation $2x^2+5x+3 = 0$,

$$x = \frac{-5 \pm \sqrt{25-24}}{4} = -\frac{5 \pm 1}{4}$$

$$= -\frac{3}{2} \text{ or } -1.$$

(3) If $b^2 < 4ac$, then the expression under the square root sign is negative, and therefore the equation has only imaginary roots.

E.g. in the equation $7x^2+5x+1 = 0$,

$$x = \frac{-5 \pm \sqrt{25-28}}{14} = \frac{-5 \pm \sqrt{-3}}{14}$$

$$= \frac{-5 \pm i\sqrt{3}}{14}$$

where $i = \sqrt{-1}$ (see p. 26).

(*c*) **Graphical Methods** (see p. 127).

Higher Equations.—All equations higher than quadratic are best solved graphically (see p. 127), unless it is possible to deal with them by the method of factorisation.

EXAMPLE ON SIMPLE EQUATIONS.

An average person loses nitrogen in his excretions corresponding to 1 gramme of protein per kilo body weight. A diabetic patient is to be supplied with 35 calories per kilo body weight a day, distributed over protein, carbohydrate and fat. Assuming that each gramme of protein yields in the body 0·6 gramme of a carbohydrate moiety and that for each 3 grammes of fat supplied in the diet there must be supplied also 1 gramme of carbohydrate for complete combustion of the fat and the avoidance of ketosis, how much of protein, carbohydrate and fat must be given per day to such a patient weighing 70 kilograms? The heat values of protein, carbohydrate and fat are 4, 4 and 9 calories per gramme respectively.

1 gramme of protein per kilo body weight must be given for nitrogen equilibrium, yielding 4 calories.

This yields 0·6 gramme carbohydrate, which can burn up 1·8 grammes fat, yielding $1{\cdot}8 \times 9 = 16{\cdot}2$ calories, making a total of about 20 calories

per kilo, and leaving a balance of 15 calories to be made up from carbohydrate and fat in the proportions by weight of 1 : 3.

Let x = weight of carbohydrate, yielding $4x$ calories.
$\therefore$ $3x$ = weight of fat, yielding $27x$ calories.
$\therefore$ $31x$ calories = 15, whence $x = 0{\cdot}5$ gramme approximately.

We therefore have

Protein 1 gramme per kilo	= 4 calories	= 35·7 calories per kilo.
Carbohydrate 0·5 gramme per kilo	= 2 ,,	
Fat = $(1{\cdot}8+1{\cdot}5) = 3{\cdot}3$ grammes	= 29·7 ,,	

$\therefore$ The 70-kilogram patient theoretically needs 70 grammes protein, 35 grammes carbohydrate and 230 grammes fat per day.

[In actual practice he is given 1 gramme protein, 1 gramme carbohydrate and about 1·6 grammes fat per kilo body weight and the combustion of carbohydrate is, if necessary, aided by means of insulin.]

EXAMPLES ON QUADRATIC EQUATIONS.

(1) In a problem on Biochemistry (see p. 265), the following equation occurs:—

$$\frac{n^2}{(1-n)(a-n)} = 4.$$

Find the value of n in terms of a.

Multiplying out we get

$$n^2 = 4a - 4an - 4n + 4n^2$$

or

$$3n^2 - 4n(a+1) + 4a = 0.$$

This is a quadratic equation in n similar in form to $ax^2+bx+c = 0$, except that the coefficients are different, viz.:

$$a = 3;\quad b = -4(a+1);\quad c = 4a.$$

Now

$$\begin{aligned} n &= \frac{1}{2a}(-b \pm \sqrt{b^2-4ac}) \\ &= \tfrac{1}{6}[4(a+1) \pm \sqrt{16(a+1)^2 - 48a}] \\ &= \tfrac{1}{6}[4(a+1) \pm 4\sqrt{a^2+2a+1-3a}] \\ &= \tfrac{1}{3}[2(a+1) \pm 2\sqrt{a^2-a+1}] \\ &= \tfrac{2}{3}(a+1 \pm \sqrt{a^2-a+1}). \end{aligned}$$

(2) Find the value of

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}} \text{ to infinity.}$$

Let the value of this expression be x.

Then

$$\begin{aligned} x^2 &= 2\sqrt{2\sqrt{2\sqrt{2\ldots}}} \text{ to infinity} \\ &= 2x. \end{aligned}$$

$$\therefore\ x^2 - 2x = 0, \quad\text{or}\quad x(x-2) = 0.$$

$$\therefore\ x = 2.$$

Similarly

$$\sqrt{n\sqrt{n\sqrt{n\sqrt{n\ldots}}}} \text{ to infinity} = n.$$

(3) Find the value of

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}\ldots}}} \text{ to infinity.}$$

Let the value of this expression be x.

Then $$x^2 = 2+x.$$

$$\therefore \quad x^2-x-2 = 0.$$

$$\therefore \quad x = \frac{1\pm\sqrt{1+8}}{2} = \frac{1\pm 3}{2}.$$

Ignoring the impossible negative value of -1, we get $x = 2$.

(4) When acetic acid, CH_3COOH, is dissolved in water, a portion of it undergoes dissociation into the ions CH_3COO^- and H^+, and the remaining portion remains undissociated, the relative amounts of dissociated and undissociated portions depending upon the degree of dilution of the acid and other factors.

Assuming dissociation to take place in accordance with the Law of Mass Action, viz. $K[CH_3COOH] = [CH_3COO^-][H^+]$, where the brackets stand for the concentration of the substance enclosed, calculate the hydrogen-ion concentration $[H^+]$ of a solution of acetic acid containing one gramme-molecule of acid in v volumes of water, it being given that K for acetic acid (at 18° C.) $= 1{\cdot}86 \times 10^{-5}$. Hence find the hydrogen-ion concentration of the acid in normal and decinormal solution. What are the respective pH values?

Assume that out of the gramme-molecule of CH_3COOH a portion a has undergone dissociation.

Then concentration of undissociated acid $= \dfrac{1-a}{v} = [CH_3COOH]$.

,, ,, dissociated portion $= \dfrac{a}{v} = [H^+]$ or $[CH_3COO^-]$

But, since the dissociated portion contains equivalent quantities of the two ions H^+ and CH_3COO^-, *i.e.* $[CH_3COO^-] = [H^+]$,

$\therefore$ $K[CH_3COOH]$ which equals $[CH_3COO^-][H^+] = [H^+]^2$

or $$K\left(\frac{1-a}{v}\right) = \left(\frac{a}{v}\right)^2$$

or $$a^2+Kva-Kv = 0.$$

$$\therefore \quad a = \frac{-Kv\pm\sqrt{K^2v^2+4Kv}}{2}.$$

As a cannot be negative, the only admissible root is

$$a = \frac{-Kv+\sqrt{K^2v^2+4Kv}}{2}.$$

But $$K = 1{\cdot}86\times 10^{-5}.$$

$$\therefore \quad a = \frac{-1{\cdot}86\times 10^{-5}v+\sqrt{(1{\cdot}86)^2\times 10^{-10}v^2+4\times 1{\cdot}86\times 10^{-5}v}}{2}.$$

As $(1{\cdot}86)^2\times 10^{-10}v^2$ is very small compared with $4\times 1{\cdot}86\times 10^{-5}v$, it may be neglected.

Hence when $v = 1$ litre, *i.e.* in *normal* solution,

$$\frac{a}{v} \text{ or } [H^+] = -0{\cdot}93 \times 10^{-5} + 10^{-3}\sqrt{18{\cdot}6} = 4{\cdot}304 \times 10^{-3},$$

and when $v = 10$ litres, *i.e.* in *decinormal* solution,

$$\frac{a}{v} \text{ or } [H^+] = \frac{-0{\cdot}93 \times 10^{-4} + 10^{-2}\sqrt{1{\cdot}86}}{10} = 1{\cdot}354 \times 10^{-3}.$$

Hence, in normal solution $4{\cdot}304 \times 10^{-3}$ grammes of H^+-ions per litre, and in decinormal solution $1{\cdot}354 \times 10^{-3}$ grammes of H^+-ions per litre are liberated.

Since $\log 4{\cdot}304 \times 10^{-3} = \log 4{\cdot}304 - 3 = 0{\cdot}6339 - 3 = -2{\cdot}37$
and $\log 1{\cdot}354 \times 10^{-3} = \log 1{\cdot}354 - 3 = 0{\cdot}1316 - 3 = -2{\cdot}87$,

$\therefore$ pH value of the normal solution $= 2{\cdot}37$, and pH value of the decinormal solution $= 2{\cdot}87$. Hence, although the difference in pH values is no more than $0{\cdot}5$, the normal solution is more than three times (*i.e.* $4{\cdot}304/1{\cdot}354$) as acid as the decinormal solution.

Simultaneous Equations, or equations with more than one unknown quantity.

The following example will illustrate a method of dealing with such equations:—

Solve
$$12x + 11y = 12 \quad . \quad . \quad . \quad \text{(i)}$$
$$42x + 22y = 40{\cdot}5 \quad . \quad . \quad . \quad \text{(ii)}$$

Multiply the 1st equation throughout by 2, thus:

$$24x + 22y = 24.$$

Subtract this from (ii) as follows:—

$$\begin{aligned} 42x + 22y &= 40{\cdot}5 \\ 24x + 22y &= 24 \\ \hline 18x \qquad &= 16{\cdot}5 \end{aligned}$$

$$\therefore \quad x = \frac{16{\cdot}5}{18} = \frac{11}{12}.$$

If now we substitute this value of x in either of the original equations (*e.g.* in (i)), we get

$$12 \times \tfrac{11}{12} + 11y = 12$$

or
$$11 + 11y = 12.$$
$$\therefore \quad 11y = 1.$$
$$\therefore \quad y = \tfrac{1}{11}.$$

Hence the solution is $x = \frac{11}{12}$, $y = \frac{1}{11}$.

Simultaneous equations can also be solved graphically (see p. 128).

Such equations occur frequently in the analysis of curves.

EXAMPLE.

How much bread, cheese and butter are required to supply 100 grammes of protein, 100 grammes of fat and 500 grammes of carbohydrate? The following are, in round numbers, the percentage compositions of those articles of diet:—

	Protein.	Fat.	Carbohydrate.
Bread . . .	8	1	50
Cheese . . .	30	20	1
Butter . . .	2	80	1

Let there be required

x grammes of bread,
y grammes of cheese,
z grammes of butter.

We then have the following equations:—

Protein $0{\cdot}08x+0{\cdot}3\ y+0{\cdot}02z = 100$ grammes . . (i)
Fat $0{\cdot}01x+0{\cdot}2\ y+0{\cdot}80z = 100$ grammes . . (ii)
Carbohydrate $0{\cdot}5\ x+0{\cdot}01y+0{\cdot}01z = 500$ grammes . . (iii)

Multiplying (ii) by 8 we get

$$0{\cdot}08x+1{\cdot}6y+6{\cdot}4\ z = 800$$

Subtract (i) $\quad 0{\cdot}08x+0{\cdot}3y+0{\cdot}02z = 100$

and get $\quad 1{\cdot}3y+6{\cdot}38z = 700$ . . . (a)

Similarly (iii) $\quad 0{\cdot}5x+\ 0{\cdot}01y+\ 0{\cdot}01z = \ 500$
(ii) × 50 $\quad 0{\cdot}5x+10{\cdot}00y+40{\cdot}00z = 5000$

By subtraction $\quad 9{\cdot}99y+39{\cdot}99z = 4500$
or in round numbers $\quad 10y+40{\cdot}00z = 4500$
i.e. $\quad y+4z = 450$ . . . . . (b)

Now from (a) we have

$$1{\cdot}3y+6{\cdot}38z = 700$$

Subtracting (b) × 1·3 $\quad 1{\cdot}3y+5{\cdot}20z = 585$

$$1{\cdot}18z = 115$$
$$\therefore\quad z = 97{\cdot}5 \text{ grammes.}$$

Substituting this value of z in (b) we get

$$y+390 = 450.$$
$$\therefore\quad y = 60 \text{ grammes.}$$

Substituting these values of y and z in (iii) we get

$$0{\cdot}5x\ +0{\cdot}6+0{\cdot}98 = 500$$
$$0{\cdot}5x = 500-1{\cdot}6 = 498{\cdot}4.$$
$$\therefore\quad x = 996{\cdot}8 \text{ grammes.}$$

∴ In round numbers there are required

1000 grammes or 2 lb. 3 oz. of bread,
60 grammes or 2·1 oz. of cheese,
and 100 grammes or 3·5 oz. of butter.

EXERCISES.

(1) How much cane sugar ($C_{12}H_{22}O_{11}$) and dry albumen (containing 15 per cent. N and 53 per cent. C) are required in a mixed dietary to furnish 30 grammes N and 350 grammes C?

[*Answer.* If x = amount of sugar and y = amount of albumen we have $0{\cdot}15y = 30$. $\therefore y = 200$. Also formula for sugar shows it to contain 42 per cent. carbon. $\therefore 0{\cdot}42x + 0{\cdot}53y = 350$ or $0{\cdot}42x + 106 = 350$. $\therefore x = \dfrac{244}{0{\cdot}42} = 581$. Hence 581 grammes of sugar and 200 grammes of albumen are required.]

(2) Find the amount of bread, meat, butter and potatoes required by a man doing an ordinary amount of work (*i.e.* to supply 4 oz. of protein, 3 oz. of fat, 16 oz. of carbohydrate), given the following compositions:—

	Protein.	Fat.	Carbohydrates.	Salts.
Beef . . .	15	8·4	..	1·6
Bread . .	8	1	50	1·5
Salt butter .	..	80	..	3·0
Potatoes . .	2	0·2	21·84	1·0

[*Answer.* 1 lb. beef, 5·2 oz. bread, 1·42 oz. butter, 3·8 lb. potatoes.]

Irrational Numbers.—All surds, as well as incommensurable numbers such as π, or e (p. 13), which cannot be expressed in the form of an integer or simple fraction, are called irrational numbers.

Imaginary Quantities.—Since the square of every number, whether positive or negative, is always positive, *e.g.* $(+a)^2 = +a^2$, and $(-a)^2 = +a^2$, therefore there is no real quantity known whose square is negative. Hence, $\sqrt{-1}$, or $\sqrt{-2}$ or $\sqrt{-a}$ can only be imaginary, since their squares produce negative quantities, thus:

$$(\sqrt{-1})^2 = -1; \quad (\sqrt{-2})^2 = -2; \quad (\sqrt{-a})^2 = -a.$$

Definition.—An imaginary quantity is the square root of a negative quantity.

Such imaginary quantities frequently occur in mathematical analysis, and it is therefore necessary to say a few words about them here.

Notation. $\sqrt{-1}$ is indicated by the letter i.

Therefore
$$\sqrt{-2} = i\sqrt{2}.$$
$$\sqrt{-3} = i\sqrt{3}.$$
$$\sqrt{-4} = i\sqrt{4} = 2i.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\sqrt{-a} = i\sqrt{a}.$$

Properties of Imaginary Numbers.—

(1) *Powers.* (i) $i^1 = i$; (ii) $i^2 = -1$;
(iii) $i^3 = i^2 \times i = -1 \times i = -i$;
(iv) $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$.

After this the results recur, viz. $i^5 = i$; $i^6 = -1$; and so on.

(2) If $a + bi = c + di$, then $a = c$ and $b = d$. For $(a - c) = (d - b)i$, and, therefore, unless $a = c$ and $b = d$, when each side $= 0$, we shall have $(a - c)$ which is real, equal to $(d - b)i$ which is imaginary.

Maxima and Minima.—The question regarding the maximum or minimum value that any algebraical expression can have is one of considerable importance to the biological student and will be dealt with in some detail in Chapter XI. This is, however, a convenient place at which to refer to the subject.

Consider the general expression for the roots of a quadratic equation, viz. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Since the square of a real quantity can never be negative or less than zero, it is clear that the least value of $\sqrt{b^2 - 4ac}$ is 0, *i.e.* when $b^2 = 4ac$. In that case $x = \dfrac{-b}{2a}$.

EXAMPLES.

(1) By the Law of Mass Action $[H^+][OH^-]$ = constant = C, *i.e.* the hydrogen-ion concentration of water multiplied by the hydroxyl-ion concentration is constant. What must be the relative concentrations of these ions in order to make their sum a minimum?

From $[H^+][OH^-] = C$, we have $[H^+] = \dfrac{C}{[OH^-]}$.

If therefore we designate $[OH^-]$ by x, and the sum of the two ionic concentrations by S, we have

$$S = x + \frac{C}{x}.$$

$$\therefore \quad x^2 - Sx + C = 0,$$

whence
$$x = \frac{S \pm \sqrt{S^2 - 4C}}{2}.$$

The least value that S can have is when $S^2 = 4C$, *i.e.* when $S = 2\sqrt{C}$. This makes $x = \frac{S}{2} = \sqrt{C}$. Hence $[OH^-]$, which equals x, $= \sqrt{C}$, and $[H^+]$, which equals $\frac{C}{[OH^-]}$, $= \frac{C}{\sqrt{C}} = \sqrt{C}$.

Hence, *the sum of the two concentrations will be a minimum when the concentrations are equal,* i.e. *at the neutral point.*

Verification.—At the neutral point $[H^+] = [OH^-] = 10^{-7}$. $\therefore$ $[H^+] + [OH^-] = 2 \times 10^{-7}$. When $[H^+]$ is, say, $10^{-6 \cdot 99}$ (*i.e.* $10^{0 \cdot 01 - 7}$), $[OH^-] = 10^{0 \cdot 99 - 8}$ (since $[H^+][OH^-]$ = constant $= 10^{-14}$). $\therefore$ $[H^+] + [OH^-] = 1{\cdot}023 \times 10^{-7} + 9{\cdot}977 \times 10^{-8} = 2{\cdot}02 \times 10^{-7}$, which is greater than 2×10^{-7}. As the divergence between $[H^+]$ and $[OH^-]$ increases so does their sum increase.

(2) The rate of hydrolysis of the ester ethyl acetate in the presence of acetic acid is proportional to the product of the concentrations (*i.e.* the number of gramme-molecules per litre) of the ester and the acid. The hydrolysis takes place in accordance with the equation

$$\underset{\text{Ethyl acetate}}{CH_3COOC_2H_5} + H_2O = \underset{\text{Acetic acid}}{CH_3COOH} + C_2H_5OH.$$

At what stage of the reaction is the rate of hydrolysis at its maximum?

Let a and b represent the initial concentrations of the ester and the acid respectively. Then, when x gramme-molecules of the ester have been hydrolysed into x gramme-molecules of acetic acid, the rate of hydrolysis at that moment is proportional to the product of the two concentrations, viz. $(a-x)(b+x)$. Our problem, therefore, is to find for which value of x, $(a-x)(b+x)$ is a maximum.

Multiplying out, the expression becomes

$$ab + (a-b)x - x^2.$$

Let this equal m.

Then $$x^2 - (a-b)x - (ab - m) = 0.$$

Whence $$x = \frac{(a-b) \pm \sqrt{(a-b)^2 + 4(ab-m)}}{2} = \frac{(a-b) \pm \sqrt{(a+b)^2 - 4m}}{2}.$$

The greatest value that m can have is when $\sqrt{(a+b)^2 - 4m} = 0$, and then $x = \frac{a-b}{2}$.

Hence, *the maximum rate of hydrolysis occurs at the moment when the number of gramme-molecules of ester hydrolysed is equal to half the difference of the initial concentrations of the ester and the acid.*

Verification.—Let the initial concentrations be $a = 1$ and $b = 0{\cdot}01$. When $x = \frac{a-b}{2} = 0{\cdot}495$ gramme-molecule, $(a-x)(b+x) = (1 - 0{\cdot}495)(0{\cdot}01 + 0{\cdot}495) = 0{\cdot}505 \times 0{\cdot}505 = 0{\cdot}255025$. When x is less than $\frac{a-b}{2}$, say when $x = 0{\cdot}494$, then $(a-x)(b+x) = 0{\cdot}506 \times 0{\cdot}504 = 0{\cdot}255024$. When x is greater than $\frac{a-b}{2}$, say $0{\cdot}496$, $(a-x)(b+x) = 0{\cdot}504 \times 0{\cdot}506 = 0{\cdot}255024$. Hence whether x is greater or less than $\frac{a-b}{2}$ the rate of hydrolysis is less than when x is equal to $\frac{a-b}{2}$.

Note.—Most of the problems on Maxima and Minima cannot be solved in this way and have to be dealt with by means of the differential calculus (see Chapter XI.).

EXERCISES.

(1) For what value of x is the expression $2x - x^2$ a maximum.

[*Answer*, 1.]

(2) Find the least possible value of $4x + \frac{9}{x}$, and the value of x in that case.

[*Answer*, Minimum = 12, when x = 1·5.]

(3) Prove that in the case of several rectangles of the same area that which has equal sides has the least perimeter. [See Example (1).]

SURDS.

Definition.—A surd is the root of an exact number which cannot be exactly determined, and which cannot therefore be expressed by an integer or by a finite fraction.

E.g. $\sqrt[2]{2}$ or $\sqrt[3]{5}$, etc.

Surds frequently occur in mathematical work, and in order to reduce the labour of calculation to a minimum, certain artifices may have to be employed.

E.g.
$$\frac{\sqrt{5}-1}{\sqrt{5}+1}.$$

If we had to find the value of this expression by first extracting the square root of 5, which is 2·2361, and then performing the various operations, the work would be laborious, thus:

$$\frac{2{\cdot}2361 - 1}{2{\cdot}2361 + 1} = \frac{1{\cdot}2361}{3{\cdot}2361}$$
$$= 0{\cdot}38195.$$

But by remembering that $(a+b)(a-b) = a^2 - b^2$, we can simplify our work very greatly by **rationalising the denominator** as follows:—

$$\frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{(\sqrt{5}-1)(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{(\sqrt{5}-1)^2}{(\sqrt{5})^2-1}$$
$$= \frac{5-2\sqrt{5}+1}{5-1} = \frac{6-2\sqrt{5}}{4}$$
$$= \frac{3-\sqrt{5}}{2}$$
$$= \frac{3-2{\cdot}236}{2}$$
$$= \frac{0{\cdot}764}{2} = 0{\cdot}382.$$

Leaving out the unessential steps the work would appear as follows:—

$$\frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{(\sqrt{5}-1)^2}{4} = \frac{3-\sqrt{5}}{2} = \frac{0{\cdot}764}{2} = 0{\cdot}382.$$

EXERCISE.

Find the value of $\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

[*Answer*, $2-\sqrt{3} = 0{\cdot}268$.]

Partial Fractions.—Whenever we have a fraction, the denominator of which consists of the product of two or more factors, then that fraction can always be expressed as the algebraical sum of a number of fractions, each of which has as its denominator only one of the factors of the original denominator. For example, we have seen that

$$\frac{2x}{x^2-1} = \frac{1}{x+1} + \frac{1}{x-1}.$$

The component fractions $\dfrac{1}{x+1}$ *and* $\dfrac{1}{x-1}$, *of which the original fraction* $\dfrac{2x}{x^2-1}$ *is composed, are called the partial fractions of* $\dfrac{2x}{x^2-1}$.

Now, whilst it is easy to add together several simple fractions to get a single more complicated fraction, the reverse process of splitting up a complicated fraction into its simpler constituents or its partial fractions is not always so easy, although it can always be done. In the higher mathematics such splitting of a fraction into its partial fractions is very often necessary (see pp. 177, 261 and 307), and we shall therefore take a few typical cases to show how, by making use of the properties of an identity, such a splitting can be effected.

Example.—Supposing we did not know what the partial fractions of $\dfrac{2x}{x^2-1}$ were. How could we set about to find them?

The first thing we have to do is to discover what are the factors of the denominator of the given fraction. In our case the denominator is x^2-1, and we know that $(x^2-1) = (x+1)(x-1)$, so that

$$\frac{2x}{x^2-1} = \frac{2x}{(x+1)(x-1)}.$$

Hence we know that our fraction must consist of two simpler fractions, the denominators of which are $(x+1)$ and $(x-1)$ respectively. The only thing, therefore, that remains to be discovered is, what are the respective numerators of these component or partial fractions?

Call these numerators A and B, and we get

$$\frac{2x}{x^2-1} = \frac{2x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}.$$

If now we perform the addition of these partial fractions in the ordinary way we get

$$\frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)};$$

and this must be equal to $\dfrac{2x}{(x+1)(x-1)}$.

$$\therefore \quad \frac{2x}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}.$$

$$\therefore \quad 2x = A(x-1)+B(x+1).$$

This being an identity is true for all values of x; let us therefore put $x=1$, when we shall get

$$2 = A(1-1)+B(1+1) = 2B.$$

$$\therefore \quad B = 1.$$

Similarly, by putting $x = -1$ we get

$$-2 = -2A$$

giving $\quad A = 1.$

$$\therefore \quad \frac{2x}{(x+1)(x-1)} \text{ which } = \frac{A}{x+1} + \frac{B}{x-1}$$

$$= \frac{1}{x+1} + \frac{1}{x-1}.$$

EXAMPLES.

(1) Find the partial fractions of

$$\frac{3x+2}{x^3-6x^2+11x-6}.$$

We first find the factors of the denominator by using the method employed in Example (1), p. 18. In that way we find that

$$x^3-6x^2+11x-6 = (x-1)(x-2)(x-3).$$

We therefore put $\dfrac{3x+2}{x^3-6x^2+11x-6} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$$= \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{x^3-6x^2+11x-6}.$$

$$\therefore \quad 3x+2 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2),$$

an identity which must therefore be true for all values of x.

By putting $x = 1$ we get

$$3+2 = A(-1)(-2)+0+0$$
$$= 2A.$$
$$\therefore \quad A = 5/2.$$

By putting $x = 2$ we get

$$B = -8$$

and by putting $x = 3$ we get

$$C = 11/2.$$

$$\therefore \quad \frac{3x+2}{x^3-6x^2+11x-6} = \frac{5}{2(x-1)} - \frac{8}{(x-2)} + \frac{11}{2(x-3)}.$$

The importance of partial fractions will be realised when we come to deal with differentiation (Chapter X., p. 177) and integration (Chapter XIX., p. 307).

(2) Resolve into partial fractions

$$\frac{x^3+3x+1}{(1-x)^4}.$$

In cases like this where the denominator contains a power of a single factor (called a *repeating factor*), we employ a very useful artifice or dodge which simplifies our operations very considerably.

We say, let

$$1-x = z.$$

$$\therefore \quad x = 1-z.$$

Hence

$$\begin{aligned}
\frac{x^3+3x+1}{(1-x)^4} &= \frac{(1-z)^3+3(1-z)+1}{z^4} \\
&= \frac{1-3z+3z^2-z^3+3-3z+1}{z^4} \\
&= \frac{5-6z+3z^2-z^3}{z^4} \\
&= \frac{5}{z^4} - \frac{6}{z^3} + \frac{3}{z^2} - \frac{1}{z} \\
&= \frac{5}{(1-x)^4} - \frac{6}{(1-x)^3} + \frac{3}{(1-x)^2} - \frac{1}{1-x}.
\end{aligned}$$

EXERCISES.

(1) Resolve into partial fractions

$$\frac{4x^2+2x-14}{x^3+3x^2-x-3}.$$

$$\left[\textit{Answer}, \ \frac{3}{x+1} - \frac{1}{x-1} + \frac{2}{x+3}.\right]$$

(2) Resolve into partial fractions

$$\frac{3x^3-5x^2+4}{(x-1)^3(x^2+1)}.$$

$$\left[\textit{Answer}, \ \frac{1}{(x-1)^3} - \frac{3}{2(x-1)^2} + \frac{3}{(x-1)} + \frac{3(1-2x)}{2(x^2+1)}.\right]$$

CHAPTER IV.

A FEW POINTS IN ELEMENTARY TRIGONOMETRY.

Trigonometry has to do with the relations between the sides of a right-angled triangle, the other angles of which are known.

The student will notice that if one of the other two angles of a right-angled triangle is known, then the remaining angle is also known.

For in any right-angled triangle ABC (fig. 1), if $\angle$ B = 90°, then $\angle$s A + C also = 90° (since the sum of the three angles = 180°).

$\therefore$ If, say, angle A is known, then C is also known, since $\angle$ C = 90° − A.

Thus if $\angle$ A = 30°, then $\angle$ C = 90° − 30° = 60°.

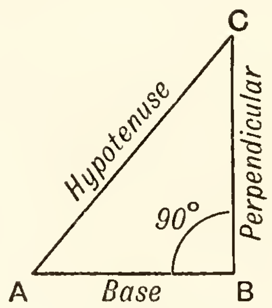

Fig. 1.—Right-angled Triangle.

The Trigonometrical Ratios.—Supposing we fix our attention on the angle A in the right-angled triangle ABC, then

(i) the side BC, which is opposite the angle A, is called the *perpendicular*;

(ii) the side AC, which is opposite the right angle, is called the *hypotenuse*;

(iii) the third side AB, which is adjacent to the right angle and the angle A, is called the *base*.

From these three sides we can form six different ratios as follows:—

(i) $\frac{BC}{AC}$ or $\frac{\text{perpendicular}}{\text{hypotenuse}}$ is called the *sine* of the angle BAC or A, and is written sin A.

(ii) $\frac{AB}{AC}$ or $\frac{\text{base}}{\text{hypotenuse}}$ is called the *cosine* of the angle BAC or A, and is written cos A.

(iii) $\frac{BC}{AB}$ or $\frac{\text{perpendicular}}{\text{base}}$ is called the *tangent* of the angle BAC or A, and is written tan A.

(iv) $\frac{AC}{BC}$ or $\frac{\text{hypotenuse}}{\text{perpendicular}}$ is called the *cosecant* of the angle BAC or A, and is written cosec A.

(v) $\frac{AC}{AB}$ or $\frac{\text{hypotenuse}}{\text{base}}$ is called the *secant* of the angle BAC or A, and is written sec A.

(vi) $\frac{AB}{BC}$ or $\frac{\text{base}}{\text{perpendicular}}$ is called the *cotangent* of the angle BAC or A, and is written cot A.

The important ratios to remember are the first three, viz. sin A, cos A and tan A. The other three ratios are formed by inverting or taking the reciprocals of the first three. Thus,

$$\operatorname{cosec} A = \frac{1}{\sin A}; \quad \sec A = \frac{1}{\cos A}; \quad \cot A = \frac{1}{\tan A}.$$

The Powers of Trigonometrical Ratios.—If we have to write the square, cube, or any other power of any of these ratios, *e.g.* the square of sin A, we cannot write sin A^2 because this would mean the sine of the angle A^2. We could write it as $(\sin A)^2$, involving the use of brackets. A simpler way of writing it, however, is $\sin^2 A$. We therefore write:

$\text{Sin}^2 A$ instead of $(\sin A)^2$, or $\sin^n A$ instead of $(\sin A)^n$,
$\text{Cos}^2 A$ instead of $(\cos A)^2$, or $\cos^n A$ instead of $(\cos A)^n$,
$\text{Tan}^2 A$ instead of $(\tan A)^2$, or $\tan^n A$ instead of $(\tan A)^n$,
and so on.

Note.—It is very important that the student should distinguish between the meanings of the terms *geometrical tangent*, which means the line touching a curve at one point only, and the *trigonometrical tangent* of an angle which, as we have just seen, means the ratio $\frac{\text{perpendicular}}{\text{base}}$ of the right-angled triangle of which the angle is the angle at the base (and opposite the perpendicular).

The following **relations** between the various ratios are most important:—

(i) *$Sin^2 A + cos^2 A = 1$*, whatever the angle A may be. This is almost obvious from fig. 1.

For $\qquad \sin^2 A = \frac{BC^2}{AC^2}$ and $\cos^2 A = \frac{AB^2}{AC^2}$.

$$\therefore \quad \sin^2 A + \cos^2 A = \frac{BC^2 + AB^2}{AC^2} = \frac{AC^2}{AC^2} = 1.$$

Corollaries.—*Sin² A = 1 − cos² A ; cos² A = 1 − sin² A.*

(ii) $Tan\ A = \dfrac{sin\ A}{cos\ A}$.

For $\qquad \sin A = \dfrac{BC}{AC}$ and $\cos A = \dfrac{AB}{AC}$.

$$\therefore \qquad \frac{\sin A}{\cos A} = \frac{BC}{AC} \div \frac{AB}{AC} = \frac{BC}{AB} = \tan A.$$

(iii) Similarly, $cot\ A = \dfrac{cos\ A}{sin\ A}$.

(iv) $Sec^2\ A = 1 + tan^2\ A$.

For $\qquad \sec^2 A = \dfrac{1}{\cos^2 A} = \dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}$ (see (i)),

$$= \frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A}$$

$$= \tan^2 A + 1.$$

(v) Similarly $cosec^2\ A = 1 + cot^2\ A$.

Inverse Ratios.—When we wish to say that A is an angle whose sine, cosine or tangent, etc., is equal to m, we write as follows:—

$A = \sin^{-1} m$; or $A = \cos^{-1} m$; or $A = \tan^{-1} m$; etc.,
or $A = \text{arc} \sin m$; $A = \text{arc} \cos m$; $A = \text{arc} \tan m$; etc.

These quantities, which are called "Inverse Circular Ratios," are read as "sine minus one m," etc., and must be distinguished from $(\sin m)^{-1}$ which means $\dfrac{1}{\sin m}$.

Use of Trigonometrical Identities for Simplification of Arithmetical Operations.—Familiarity with a couple of trigonometrical identities occasionally affords one a ready means for greatly reducing the labour involved in some complicated arithmetical operations. The identities which most readily lend themselves to such use are

$$(1)\ \sin^2 \theta = 1 - \cos^2 \theta,$$
$$(2)\ \sec^2 \theta = 1 + \tan^2 \theta.$$

By means of the first of these identities a difference of two

squares is reduced to one squared number, and by means of the second a sum of two squares is reduced to one squared number.

EXAMPLES.

(1) Find the value of $\dfrac{\sqrt{1-(0{\cdot}6202)^2}}{0{\cdot}6202}$.

The trigonometrical tables give 0·6202 as the cosine of 51° 40′.

$\therefore$ Expression becomes

$$\frac{\sqrt{1-\cos^2 51^\circ\,40'}}{\cos 51^\circ\,40'} = \frac{\sin 51^\circ\,40'}{\cos 51^\circ\,40'} = \tan 51^\circ\,40' = 1{\cdot}2647.$$

(2) Find the value of $\dfrac{1}{\sqrt{11{\cdot}78^2+5{\cdot}67^2}}$.

The denominator is the same as $11{\cdot}78\sqrt{1+\left(\dfrac{5{\cdot}67}{11{\cdot}78}\right)^2}$.

Now $\dfrac{5{\cdot}67}{11{\cdot}78} = 0{\cdot}4815$, which is found from the tables to be tan 25° 42′.

$\therefore$ Expression becomes

$$\begin{aligned}\frac{1}{11{\cdot}78\sqrt{1+\tan^2 25^\circ\,42'}} &= \frac{1}{11{\cdot}78\sec 25^\circ\,42'}\\ &= \frac{\cos 25^\circ\,42'}{11{\cdot}78}\\ &= \frac{0{\cdot}9011}{11{\cdot}78} = 0{\cdot}076.\end{aligned}$$

(3) Prove that

$$\frac{\tan A+\cot A}{\tan A-\cot A} = \frac{1}{1-2\cos^2 A}$$

$$\begin{aligned}\tan A+\cot A &= \frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\\ &= \frac{\sin^2 A+\cos^2 A}{\sin A\cos A} = \frac{1}{\sin A\cos A}\\ \tan A-\cot A &= \frac{\sin A}{\cos A}-\frac{\cos A}{\sin A}\\ &= \frac{\sin^2 A-\cos^2 A}{\sin A\cos A} = \frac{\sin^2 A+\cos^2 A-2\cos^2 A}{\sin A\cos A}\\ &= \frac{1-2\cos^2 A}{\sin A\cos A}\end{aligned}$$

$$\begin{aligned}\therefore\quad \frac{\tan A+\cot A}{\tan A-\cot A} &= \frac{1}{\sin A\cos A} \div \frac{1-2\cos^2 A}{\sin A\cos A}\\ &= \frac{1}{1-2\cos^2 A}.\end{aligned}$$

(4) Prove that $\cos^4 A - \sin^4 A = 2\cos^2 A - 1$

$$\begin{aligned}(\cos^4 A - \sin^4 A) &= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)\\ &= (\cos^2 A - \sin^2 A)\\ &= 2\cos^2 A - (\sin^2 A + \cos^2 A)\\ &= 2\cos^2 A - 1.\end{aligned}$$

EXERCISES.

Prove the following identities:—

(1) $\tan A \,.\, \operatorname{cosec} A = \sec A.$

(2) $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}.$

On the Measurement of Angles.—In the same way as logarithms are used to one of two bases, viz. 10 or e ($=2{\cdot}71828\ldots$), so are angles measured in one of two ways, viz.:

(i) The rectangular measure—with the *right angle* as the unit.
(ii) The circular measure—with the *radian* as the unit.

Definition of Radian.—If O is the centre of the semicircle ABC (fig. 2), and the arc AB is equal in length to the radius OA or OB, then the angle AOB is called a *radian*.

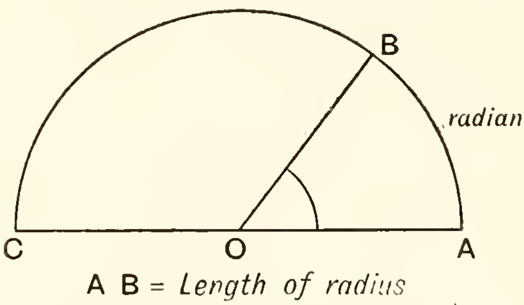

FIG. 2.—The Radian or Circular Unit of an Angle.

Whilst the rectangular measure, which gives angles in degrees, etc., is used in all numerical trigonometrical calculations, the circular measure is used in every theoretical trigonometrical analysis, *e.g.* for calculating the various trigonometrical ratios, etc. (see p. 229), because, as a length, a radian can be compared and measured with any other linear measurement. A degree, however, can no more be compared with any linear measurement than can a pound weight with a yard. The two measures can, however, be converted into each other.

The Divisions of a Right Angle.—Each right angle contains 90 degrees (or 90°). Each degree contains 60 minutes (or 60′), and each minute contains 60 seconds (or 60″).

An angle 68° 15′ 36·7″ means an angle containing 68 degrees, 15 minutes and 36·7 seconds.

To Convert an Angle from one Measurement into Another.—The whole length of the circumference of a circle $= 2\pi r$, where r = length of radius and $\pi = 3{\cdot}14159$ or approx. $\frac{22}{7}$.

But a radian is the angle measured by the arc whose length = length of radius (r).

$\therefore$ The whole length of the circumference subtends 2π radians. Also the whole circumference subtends four right angles = 360°.

$$\therefore \quad 2\pi \text{ radians} = 360°.$$

$$\therefore \quad \text{Radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = 180 \times \frac{7}{22}$$
$$= 57{\cdot}296°$$
$$= 57°\ 17'\ 45{\cdot}6''.$$

Conversely, 4 right angles = 2π radians.

$$\therefore \quad \text{A right angle} = \frac{\pi}{2} \text{ radians}$$

and $$1° = \frac{\pi}{180} \text{ radians} = 0{\cdot}01745 \text{ radian.}$$

Hence, by calculating a table of trigonometrical ratios of angles expressed in radians, one can easily convert them into rectangular measure by multiplying these ratios by $\frac{\pi}{180}$, in the same way as one can reconvert Napierian into common logarithms by multiplying by the modulus.

Note.—The student will observe that π has a double meaning. When referring to angles it stands for 2 right angles or 180°. In all other cases it represents the number 3·14159 . . . or approx. 22/7. Indeed when it refers to angles it stands for 22/7 radians (which are 180°), but the word radian is omitted.

Angular Velocity.—If a wheel makes 75 turns per minute, this means that it makes 1·25 turns per second. This again means that any point on the circumference moves through an angle of $360° \times 1{\cdot}25 = 450°$ in one second. But 360° = 6·283 radians, therefore 450° or $360° \times \frac{5}{4} = 6{\cdot}283 \times \frac{5}{4} = 7{\cdot}854$ radians. Hence we say that the **angular velocity, *i.e.* the number of radians described by the point per unit of time,** is 7·854 radians per second.

Simple Harmonic Motion.—Let a particle P starting from A move uniformly round the circumference of a circle, of radius r, in the direction indicated by the arrow (*i.e.* anticlockwise) (fig. 3). Imagine also that while P is moving on the right semi-circumference it is being illuminated by parallel rays of light falling upon it from the right, and when moving on the left semi-circumference it is illuminated by parallel light

falling upon it from the left. A shadow of P (called the projection of P upon the diameter) will then be thrown upon the diameter CD at M, and with each change of position of P upon the circumference there will be a corresponding change of position of M upon the diameter. Similarly, if the shadow of P be projected upon the diameter AB at N, there will be a corresponding change in the position of N with each movement of P. The motion of M or N along its respective diameter is a simple harmonic motion.

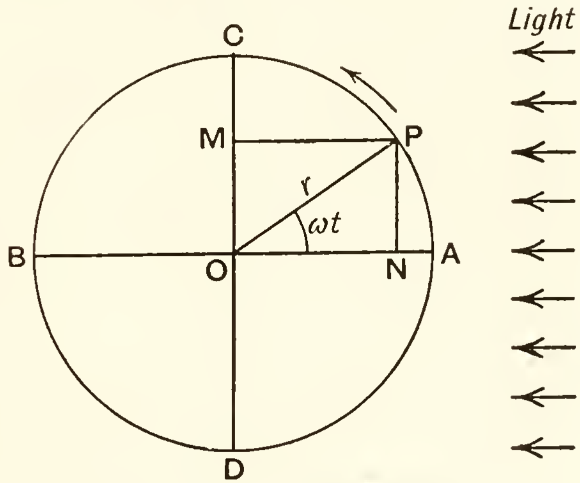

FIG. 3.—Harmonic Motion.

Definition.—*A simple harmonic motion (S.H.M.) is the motion of the projection upon a diameter of a point moving uniformly in a circle.*

Let ω = angular velocity of P.

Then angle AOP = ωt (where t = time taken for P to move from A to P). And the distances of M and N from the centre O, will be given by

$$\text{OM} = y = r \sin \omega t$$
$$\text{and} \quad \text{ON} = x = r \cos \omega t.$$

The maximum distance of either projection from the centre is called the *amplitude* and is equal to r.

The *period* or *periodic time* (T) of the S.H.M. is the time taken by N to pass from A to B and back again. $\text{T} = \dfrac{2\pi}{\omega}$.

The *frequency* (f) of the vibrations is the reciprocal of the periodic time, so that $f = \dfrac{1}{\text{T}} = \dfrac{\omega}{2\pi}$.

The Trigonometrical Ratios of Certain Angles.—In the same way as logarithm tables give the logarithms of all the numbers from 1 upwards to a certain number of figures, so do trigonometrical tables give the values of the sines, cosines, tangents, etc., of all the angles from 0° to 90° (see Appendix). The method of finding these values does not concern us here, though it will be dealt with in a later chapter. There are, however, a few angles the trigonometrical ratios of which can easily be found.

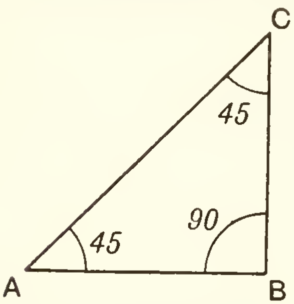

Fig. 4.—Isosceles Right-angled Triangle.

To find sin 45°; **cos** 45°; **tan** 45° (fig. 4).

$$\begin{aligned}
&\text{If} & A &= 45^\circ \text{ then C must } = 45^\circ. \\
&\therefore & AB &= BC. \\
&\text{But} & AC^2 &= AB^2 + BC^2 \\
& & &= 2AB^2 \text{ or } 2BC^2. \\
&\therefore & AC &= AB\sqrt{2} \text{ or } BC\sqrt{2}.
\end{aligned}$$

$$\therefore \quad \sin A \text{ which} = \frac{BC}{AC} = \frac{BC}{BC\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\cos A \text{ which} = \frac{AB}{AC} = \frac{\sqrt{2}}{2},$$

$$\tan A \text{ which} = \frac{BC}{AB} = 1.$$

$$\therefore \quad \sin 45^\circ = \frac{\sqrt{2}}{2}; \quad \cos 45^\circ = \frac{\sqrt{2}}{2}; \quad \tan 45^\circ = 1.$$

Similarly it can be easily proved that

$$\sin 60^\circ = \frac{\sqrt{3}}{2}; \quad \cos 60^\circ = \frac{1}{2}; \quad \tan 60^\circ = \sqrt{3};$$

$$\sin 30^\circ = \frac{1}{2}; \quad \cos 30^\circ = \frac{\sqrt{3}}{2}; \quad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

$$\begin{aligned}
\text{Also,} \quad & \sin 90^\circ = 1; \quad \cos 90^\circ = 0; \quad \tan 90^\circ = \infty \\
& \sin 0^\circ = 0; \quad \cos 0^\circ = 1; \quad \tan 0^\circ = 0.
\end{aligned}$$

The Trigonometrical Ratios of the Complement and Supplement of an Angle θ.

(*a*) The *complement* of θ is $(90^\circ - \theta)$, and it is seen from fig. 4 that if angle at A $= \theta$, then angle at C $= (90^\circ - \theta)$.

But $\sin\theta$ (*i.e.* $\sin A$) $= \frac{BC}{AC}$, and $\cos(90° - \theta)$, *i.e.* $\cos C$, $= \frac{BC}{AC}$.

$\therefore \quad \sin\theta = \cos(90° - \theta)$,

i.e. the sine of any angle is equal to the cosine of its complement.

Similarly $\tan\theta = \cot(90° - \theta)$, and $\sec\theta = \operatorname{cosec}(90° - \theta)$.

Indeed *co*sine, etc., means "the sine, etc., of the complement."

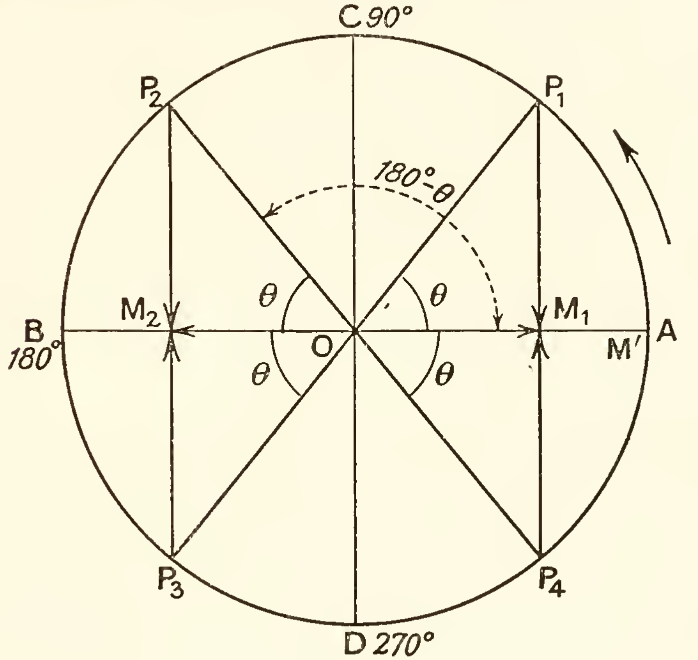

FIG. 5.

(*b*) The *supplement* of θ is $(180° - \theta)$, and it is seen from fig. 5 that:

(i) $\sin(180° - \theta)$, which is $\frac{P_2M_2}{OP_2}$, is equal to $\frac{P_1M_1}{OP_1} = \sin\theta$ (since the triangles M_1P_1O and M_2P_2O are equal in every respect).

(ii) $\cos(180° - \theta)$, which is $\frac{OM_2}{OP_2}, = -\frac{OM_1}{OP_1} = -\cos\theta$ (since $OM_2 = -OM_1$).

(iii) $\tan(180° - \theta)$, which is $\frac{P_2M_2}{OM_2}, = -\frac{P_1M_1}{OM_1} = -\tan\theta$.

Similarly,

(iv) $\sec(180° - \theta) = -\sec\theta$.

(v) $\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$.

Further, it may be seen from the figure that

$$\sin(180° + \theta), \text{ viz. } \frac{P_3M_2}{OP_3}, = -\sin\theta,$$
$$\cos(180° + \theta) = -\cos\theta,$$
$$\tan(180° + \theta) = \tan\theta.$$

Finally, $sin\,(360° - \theta)$, viz. $\frac{P_4M_1}{OP_4}$, $= -sin\,\theta$,

$cos\,(360° - \theta) = cos\,\theta$,

$tan\,(360° - \theta) = -tan\,\theta$.

EXAMPLES ON TRIGONOMETRICAL EQUATIONS.

(θ is to be taken as less than a right angle.)

(1) Find θ from the equation $4 \sin\theta = \operatorname{cosec}\theta$.

$$\operatorname{Cosec}\theta = \frac{1}{\sin\theta}.$$

$$\therefore \quad 4\sin\theta = \frac{1}{\sin\theta},$$

$$i.e.\ 4\sin^2\theta = 1.$$

$$\therefore \quad \sin\theta = \pm\tfrac{1}{2}.$$

The value of θ which makes $\sin\theta = -\frac{1}{2}$ is greater than a right angle.

$\therefore$ We take only $\sin\theta = \frac{1}{2}$.

Whence $\theta = 30°$.

(2) Find θ from the equation $\tan\theta = 3\cot\theta$.

$$\operatorname{Cot}\theta = \frac{1}{\tan\theta}.$$

$$\therefore \quad \tan\theta = \frac{3}{\tan\theta}.$$

$$\therefore \quad \tan^2\theta = 3.$$

$$\therefore \quad \tan\theta = \pm\sqrt{3}.$$

Ignoring $\tan\theta = -\sqrt{3}$, we have $\tan\theta = \sqrt{3}$.

$$\therefore \quad \theta = 60°.$$

(3) Find θ from the equation $\cos^2\theta + 2\sin^2\theta - \frac{5}{2}\sin\theta = 0$.

$$\operatorname{Cos}^2\theta = 1 - \sin^2\theta.$$

$\therefore$ Equation becomes

$$\sin^2\theta - \tfrac{5}{2}\sin\theta + 1 = 0.$$

$$\therefore \quad \sin\theta = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2}$$ (see p. 20)

$$= \tfrac{1}{2} \text{ or } 2, \text{ whence } \theta = 30°.$$

The value 2 is impossible since $\sin\theta$ cannot be greater than 1.

(4) Solve $\sec^2\theta = \tan\theta + 2{\cdot}268$.

Equation is the same as $1 + \tan^2\theta = \tan\theta + 2{\cdot}268$,

or $\tan^2\theta - \tan\theta - 1{\cdot}268 = 0$,

whence $$\tan\theta = \frac{1 \pm \sqrt{1 + 4 \times 1{\cdot}268}}{2}$$

$$= 1{\cdot}732 \text{ or } -0{\cdot}732.$$

As a negative tangent must be that of an angle greater than 90°, it is to be ignored.

$$\therefore \quad \tan\theta = 1{\cdot}732 \quad \text{and} \quad \theta = 60°.$$

Note.—So long as one has to deal with equations containing trigonometrical functions of angles only, solution by algebraical methods is fairly simple, unless those functions are of a higher degree than the second, when it is best to resort to graphical methods. In cases of equations containing an angle together with one or more of its functions, *e.g.* $\theta - \sin\theta = a$, the only solution possible is by graphical means (see p. 131, Example (3)).

EXERCISES.

(1) Simplify $\cos^4 A + 2\sin^2 A \cos^2 A$.

[*Answer,* The expression $= (\cos^2 A + \sin^2 A)^2 - \sin^4 A$
$= 1 - \sin^4 A$.]

(2) Solve $\sin\theta + \cos\theta = \sqrt{2}$. [*Answer,* $\theta = 45°$.]

(3) Solve $8\cos^2\theta - 8\cos\theta + 1 = 0$.

$$\left[\textit{Answer,}\ \cos\theta = \frac{2 \pm \sqrt{2}}{4} = 0{\cdot}8535 \text{ or } 0{\cdot}1465. \quad \therefore \quad \theta = 31°\,24' \text{ or } 81°\,35'.\right]$$

Angle of Pull of Muscle.—The amount of work performed by a muscle depends upon a number of factors, viz.:

(1) The number of contracting fibres.
(2) Their arrangement.
(3) Their degree of contraction.
(4) The angle they make with the bone to be moved.
(5) Pull per fibre.

The angle of pull keeps on altering as the bone keeps on moving.

Thus, if A is the origin of a muscle, B is its insertion, and O the joint between the two bones (figs. 6 and 7), it is seen that the angle of pull ABO keeps on altering as the bone OB, or the insertion B, moves upwards from B_1 to B_4. Now, the effective component of the contractile force F of the muscle is the component BC acting perpendicularly to the moving bone $= F\sin\theta$.

Hence, as the bone OB gets pulled up and θ increases from θ_1 to θ_2, the vertical component increases from B_1C_1 to B_2C_2. When the moving bone is in such a position OB_3 that θ_3 **is a right angle,** then the whole of the force of the muscle is spent in moving the bone, since B_3C_3 then coincides with AB_3. When OB gets pulled up still further to OB_4, θ_4 becomes greater than a right angle and the contractile force of the muscle becomes

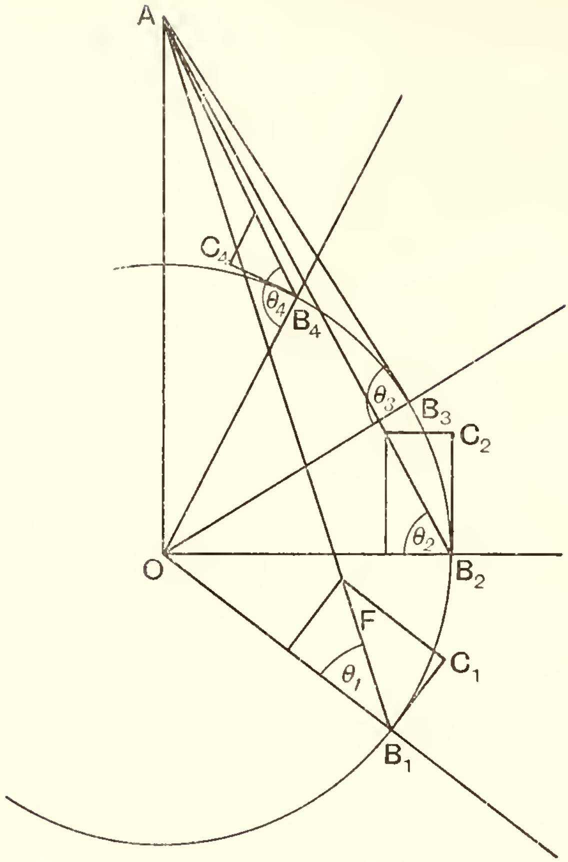

FIG. 6.—Alteration in Angle of Pull of Muscle during Contraction.

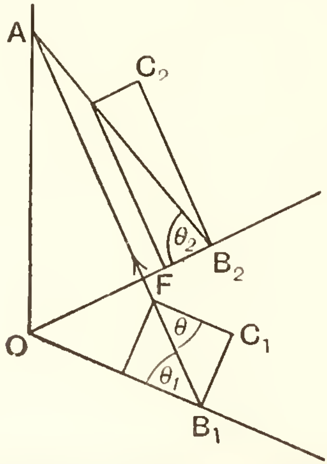

FIG. 7.—Showing Increase in Value of the Effective Component BC with Increase of Angle of Pull of Muscle (up to a right angle).

resolvable again into a vertical and a horizontal force, so that the effective pull of the muscle begins to decrease.

The maximum force of the muscle upon the moving bone is therefore obtained when $\theta = 90°$. We then have

$$F \sin \theta = F \sin 90° = F.$$

Similarly, the minimum force of the muscle exerted upon the bone is when OB is parallel to the axis of the muscle; θ is then $0°$ and $F \sin \theta = F \sin 0° = 0$.

Force of Muscle.—The force which a muscle exerts in pulling its insertion towards its origin depends, amongst other things, upon the direction of its fibres.

(*a*) **Direct Prismatic Muscle** (*e.g.* Masseter, etc.).—In muscles of this type (fig. 8), the fibres are all rectilinear, parallel to one another, and are attached at right angles to the lines of origin and insertion. (AB = origin, CD = insertion.)

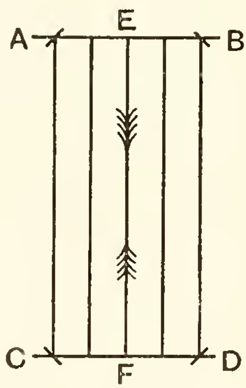

Fig. 8.—Arrangement of Fibres in Direct Prismatic Muscle.

Let f = contractile force of each fibre (of which $\frac{f}{2}$ may be considered to pull AB towards CD, and the other half, $\frac{f}{2}$, pulls CD towards AB).

Let n = number of fibres and F = total force of muscle.

Then clearly $F = nf$ (of which half, viz. $\frac{F}{2}$ or $\frac{nf}{2}$, acts in the **direction** towards CD and the other half, $\frac{F}{2}$ or $\frac{nf}{2}$, acts in the **direction** towards AB).

Also, the **line** of action of F will be EF where E and F are the middle points of AB and CD respectively.

(*b*) **Rhomboid Muscles** (*e.g.* Intercostals, Rhomboids, etc.).—In this group of muscles the fibres are also rectilinear and parallel, but they are attached obliquely to the lines of origin and insertion (fig. 9).

Here, again, $F = nf$, acting in the line EF. But as EF is oblique to AB and CD let its inclination to each of these lines be θ, and let each half of this resultant force F be split up into two components, viz.:

(1) Ek and Fm acting along the lines of origin and insertion but in opposite directions—tending to make the muscle prismatic.

(2) EL and FN acting perpendicularly to the lines of origin and insertion and pulling the origin and insertion towards each other.

We then have
$$Ek = Fm = \frac{F}{2}\cos\theta,$$
$$EL = FN = \frac{F}{2}\sin\theta.$$

Hence, total force tending to make muscle prismatic $= F\cos\theta$, and total force tending to pull the origin and insertion towards each other $= F\sin\theta$.

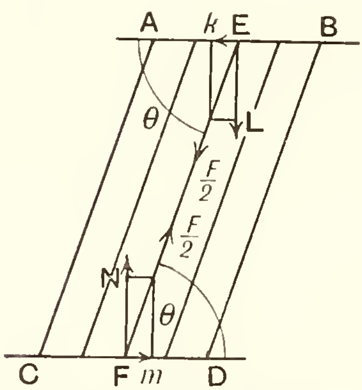

Fig. 9.—Arrangement of Fibres in Rhomboid Muscle.

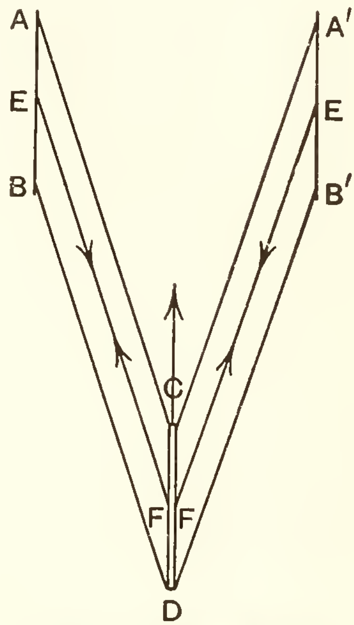

Fig. 10.—Arrangement of Fibres in Penniform Muscle.

(*c*) **Penniform Muscles** (*e.g.* Mylohyoid, Accelerator Urinæ, etc.).—In muscles of this type the arrangement of the fibres is as shown in the diagram (fig. 10).

AB and A′B′ are two lines of origin and CD is the line of insertion.

It will be seen, therefore, that such a muscle consists practically of two rhomboid arrangements symmetrically situated with regard to the line of insertion.

The total force pulling CD up in the direction of the arrow due to the action of each half of the muscle $= F\cos\theta$ (see *b* (2), above).

$\therefore$ Resultant force $R = 2F \cos \theta$.

(*d*) For **Fan-shaped Muscles,** see p. 286.

PROBLEMS LEADING TO TRIGONOMETRICAL EQUATIONS.

(1) The hand holds a weight of 10 lb. at a distance of 12 in. from the elbow. The forearm is inclined at 45° to the horizontal, and in this position the angle of pull of the biceps is 75°. What is the force with which the biceps must pull in order to hold the weight (assuming the muscle to be inserted at a distance of 2 in. from the elbow).

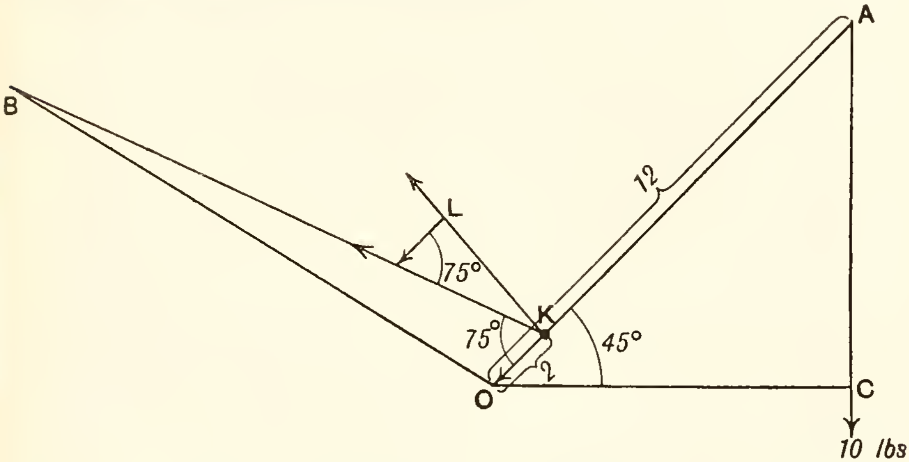

FIG. 11.—Calculation of Effective Component KL of Biceps Muscle BK.

The effective component KL of the force F of the muscle = F sin 75° (see fig. 11).

$\therefore$ By the principle of levers

$$F \sin 75° \times OK = AC \times OC = 10 \times OA \cos 45°$$

$$= 10 \times 12 \times \frac{\sqrt{2}}{2}$$

$$= 60\sqrt{2} = 84{\cdot}84 \text{ lb.}$$

$$\therefore \quad F = \frac{84{\cdot}84}{OK \sin 75°} = \frac{84{\cdot}84}{2 \sin 75°} = \frac{42{\cdot}42}{0{\cdot}9659} = 43{\cdot}9 \text{ lb.}$$

(2)—(*a*) Find the work done by the contraction of a penniform muscle. (*b*) The angle made by the fibres of the mylohyoid with the central raphé is 45°. If the longest fibre contracts by $\frac{1}{10}$ inch, how far will the middle point of the hyoid bone be drawn up?

(*a*) Work is measured by force multiplied by the distance through which the force acts in its own direction.

Now, fixing our attention on the right side of the penniform muscle, let DB (fig. 12) represent one fibre, which by contraction pulls the point B up to C. The fibre DB will therefore have contracted to DC.

Now, if CE is dropped perpendicular to DB then BE is equal to the

amount of shortening of DB (since DE is very nearly = DC), and BC is the distance through which B has been moved.

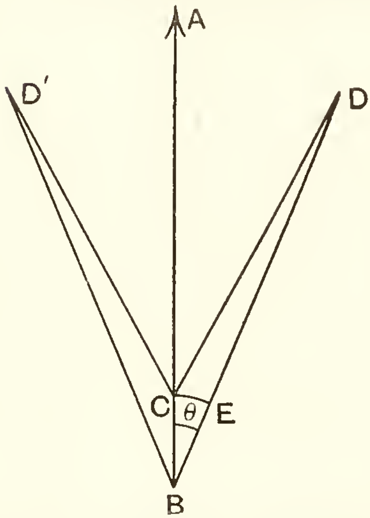

FIG. 12.—Calculation of Work of Contraction of Penniform Muscle.

But $\quad BC = \dfrac{BE}{\cos\theta} = BE \sec\theta,$

and the total force exerted by both sides of the muscle in pulling B in the direction BC is the resultant $2F\cos\theta$ (see (*c*), p. 46).

$$\therefore \quad \text{Work done} = 2F\cos\theta \,.\, BE\sec\theta$$
$$= 2F \,.\, BE.$$

But 2F . BE is the work inherent in all the muscular fibres of both sides of the muscle if arranged in a prismatic manner.

∴ Work done by penniform muscle = that done by prismatic muscle of the same length of fibre.

(*b*) From formula

$$BC = BE\sec\theta,$$

we have $\quad BC = \frac{1}{10}\sec 45° = \frac{1}{10}\sqrt{2}$

$$= 0{\cdot}14 \text{ inch.}$$

(3) In a Hüfner's spectrophotometer, the extinction coefficient is given by the formula $e = -\log\cos^2\phi$, where ϕ is the angle through which a Nicol's prism has to be rotated to restore equality of spectra. Find e, when $\phi = 61° \ 52'$.

$$\begin{aligned}\text{Log}\cos^2\phi = 2\log\cos\phi &= 2\log\cos 61° \ 52' \\ &= 2\times(\bar{1}{\cdot}6735) \text{ (from table)} \\ &= 2\times 0{\cdot}6735 - 2 \\ &= 1{\cdot}347 - 2 \\ &= -0{\cdot}653.\end{aligned}$$

$$\therefore \quad e = -2\log\cos\phi = 0{\cdot}653.$$

(4) From the laws of dynamics it is known that the distance covered by a projectile thrown into the air with a velocity of v ft. per sec. and at an angle a with the horizontal is equal to $\dfrac{v^2 \sin 2a}{g}$, where g = acceleration due to gravity. At what angle must one jump in order to cover the greatest distance?

Since the distance is proportional to $\sin 2a$,
∴ The greatest distance will be attained when $\sin 2a$ is a maximum.

But the sine of an angle increases from zero at 0° until it reaches 1 at 90°, after which it decreases again until it reaches zero at 180°.

∴ The maximum distance will be covered when $2a = 90°$, or $a = 45°$.
∴ The person must project himself at an angle of 45° with the horizontal.

The Trigonometrical Ratios of the Sum or Difference of Two Angles.—The following identities, in virtue of their very frequent occurrence in mathematical work, are important:—

$$\text{Sin}\,(A+B) = \sin A \cos B + \cos A \sin B \quad . \quad (1)$$

Whence
$$\text{Sin}\,2A = 2 \sin A \cos A \quad . \quad . \quad . \quad (1a)$$

$$\text{Sin}\,(A-B) = \sin A \cos B - \cos A \sin B \quad . \quad (2)$$

$$\text{Cos}\,(A+B) = \cos A \cos B - \sin A \sin B \quad . \quad (3)$$

Whence
$$\text{Cos}\,2A = \cos^2 A - \sin^2 A \quad . \quad . \quad . \quad (3a)$$

$$\text{Cos}\,(A-B) = \cos A \cos B + \sin A \sin B \quad . \quad (4)$$

$$\text{Tan}\,(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad . \quad . \quad . \quad (5)$$

Whence
$$\text{Tan}\,2A = \frac{2 \tan A}{1 - \tan^2 A} \quad . \quad . \quad . \quad . \quad (5a)$$

$$\text{Tan}\,(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad . \quad . \quad . \quad (6)$$

It is not necessary to give a complete proof of each of these identities, but it may be well if the student will study the proof of the first of these formulæ as a type.

Let MCN $= \angle A$ and NCK $= \angle B$ (fig. 13).

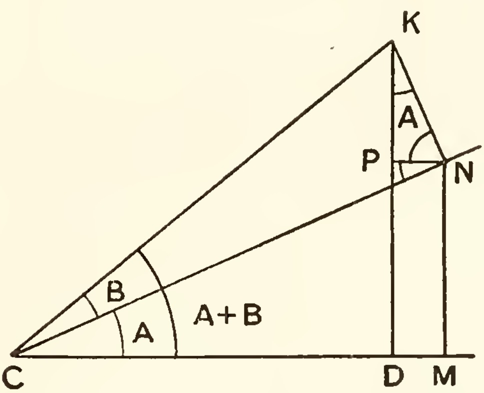

FIG. 13.—To show the Trigonometrical Ratios of the Sum of Two Angles.

Drop NM $\perp^r$ to CM. Draw NK $\perp^r$ to CN meeting CK in K. Drop KD $\perp^r$ to CM and draw NP $\parallel$ DM meeting KD in P. Then

$$\sin\,(A+B) = \frac{KD}{CK} = \frac{KP+PD}{CK} = \frac{KP+NM}{CK} = \frac{KP}{CK} + \frac{NM}{CK}.$$

Now, $\dfrac{KP}{CK} = \dfrac{KP\,.\,KN}{KN\,.\,CK}$; and $\dfrac{NM}{CK} = \dfrac{NM\,.\,CN}{CN\,.\,CK}$.

But $\dfrac{KP}{KN} = \cos PKN$, *i.e.* $\cos A$, and $\dfrac{KN}{CK} = \sin B$.

$\therefore \quad \dfrac{KP}{CK} = \cos A \sin B.$

Also $$\frac{NM}{CN} = \sin A,$$

and $$\frac{CN}{CK} = \cos B.$$

$$\therefore \quad \frac{NM}{CK} = \sin A \cos B.$$

$$\therefore \quad \sin(A+B) \text{ which } = \frac{KP}{CK} + \frac{NM}{CK}$$
$$= \cos A \sin B + \cos B \sin A.$$

Note.—From the formula $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ it follows that if $A = 45°$ or $\dfrac{\pi}{4}$, then

$$\tan\left(\frac{\pi}{4} + B\right) = \frac{\tan 45° + \tan B}{1 - \tan 45° \tan B} = \frac{1 + \tan B}{1 - \tan B}.$$

Important Corollaries to the Sum and Difference Formulæ.

By adding identities (1) and (2) on p. 49, we get

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

If we put $(A+B) = P$
and $(A-B) = Q,$

we get $$A = \frac{P+Q}{2}$$

and $$B = \frac{P-Q}{2}.$$

$$\therefore \quad \sin P + \sin Q = 2 \sin\frac{(P+Q)}{2} \cos\frac{(P-Q)}{2} \quad . \quad (a)$$

Similarly $$\sin P - \sin Q = 2 \cos\frac{(P+Q)}{2} \sin\frac{(P-Q)}{2} \quad . \quad (b)$$

$$\cos P + \cos Q = 2 \cos\frac{(P+Q)}{2} \cos\frac{(P-Q)}{2} \quad . \quad (c)$$

$$\cos P - \cos Q = -2 \sin\frac{(P+Q)}{2} \sin\frac{(P-Q)}{2} \quad . \quad (d)$$

or $$\cos Q - \cos P = 2 \sin\frac{P+Q}{2} \sin\frac{P-Q}{2} \quad . \quad . \quad (e)$$

These identities are of very great importance because not only are algebraic sums converted by their help into products

which are amenable to logarithmic computation, but they come in very useful in various trigonometrical manipulations.

EXAMPLES.

(1) Simplify $\dfrac{\sin \theta + \sin 2\theta + \sin 3\theta + \sin 4\theta}{\cos \theta + \cos 2\theta + \cos 3\theta + \cos 4\theta}$.

The expression $= \dfrac{(\sin \theta + \sin 4\theta) + (\sin 2\theta + \sin 3\theta)}{(\cos \theta + \cos 4\theta) + (\cos 2\theta + \cos 3\theta)}$,

$$= \frac{2 \sin \frac{5\theta}{2} \cos \frac{3\theta}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{\theta}{2}}{2 \cos \frac{5\theta}{2} \cos \frac{3\theta}{2} + 2 \cos \frac{5\theta}{2} \cos \frac{\theta}{2}},$$

$$= \frac{2 \sin \frac{5\theta}{2}\left(\cos \frac{3\theta}{2} + \cos \frac{\theta}{2}\right)}{2 \cos \frac{5\theta}{2}\left(\cos \frac{3\theta}{2} + \cos \frac{\theta}{2}\right)},$$

$$= \tan \frac{5\theta}{2}.$$

(2) The distance that a boy can throw a cricket ball up a certain hillside is $200 \sin x \cos (x + 30°)$ yards, where x is the angle which the direction of motion of the ball makes with the hillside when the ball leaves his hand. What must be the value of x in order to make that distance the greatest possible, and what will that distance be?

If in formula (b) on p. 50, viz.:

$$2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2} = \sin P - \sin Q,$$

we put $\dfrac{P+Q}{2} = x + 30°$,

and $\dfrac{P-Q}{2} = x$,

we obtain $P = 2x + 30°$,
and $Q = 30°$.

$\therefore \quad 2 \sin x \cos (x + 30°) = \sin (2x + 30°) - \sin 30°.$

$\therefore \quad 200 \sin x \cos (x + 30°) = 100 \sin (2x + 30°) - 50$ (since $\sin 30° = \frac{1}{2}$).

$\therefore$ For a maximum, we must have $\sin (2x + 30°)$ equal to a maximum, *i.e.* $\sin (2x + 30°) = 1 = \sin 90°$, and $\therefore 2x + 30° = 90°$, $\therefore x = 30°$, and then $100 \sin (2x + 30°) - 50 = 100 - 50 = 50$ yds.

$\therefore$ The required angle is $30°$ and the maximum distance is 50 yds.

CHAPTER V.

A FEW POINTS IN ELEMENTARY MENSURATION.

THE relation between area and volume is a matter of very great importance in many physiological problems, and it is therefore desirable for the student to refresh his memory with regard to the areas and volumes of a few of the commoner geometrical figures.

Area of a Parallelogram ABCD = DC . AE (fig. 14) (where AE is the perpendicular dropped from A to DC).

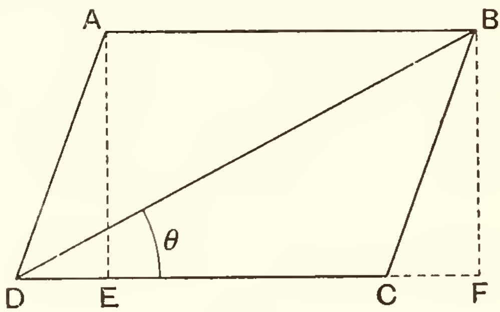

FIG. 14.

(*a*) Hence, *area of a rectangle AEFB = AB . AE, i.e. area of rectangle = product of the two sides.*

(*b*) Further, if the four sides of the rectangle are equal the rectangle becomes a *square*, and its *area* = a^2 where a = length of one of the sides.

Area of a Triangle DBC = $\frac{1}{2}$ DC . BF (where BF is the perpendicular dropped from the apex to the base DC or to the base DC produced) (fig. 14).

(*a*) If two sides DC and DB, as well as their included angle D, are known, then,

$$\text{since} \qquad \frac{BF}{DB} = \sin D, \quad \text{or} \quad BF = DB \sin D,$$

$$\therefore \quad \textit{area of triangle} = \tfrac{1}{2} DC \,.\, DB \sin D,$$

i.e. area of a triangle = half the product of two sides and the sine of the included angle.

(*b*) **Area of Right-angled Triangle** BDF (fig. 14) = $\frac{1}{2}$ DF . BF.

(*e*) **Area of Equilateral Triangle** ABC (fig. 15)

$$= \tfrac{1}{2}BC \,.\, AB \sin B,$$
$$= \tfrac{1}{2}BC^2 \sin 60°,$$
$$= \tfrac{1}{2}BC^2 \,.\, \frac{\sqrt{3}}{2} = BC^2\frac{\sqrt{3}}{4}.$$

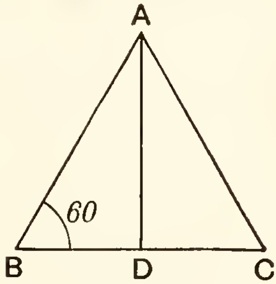

FIG. 15.

Area of a Circle = πr^2 (fig. 16) (see p. 295), where r = radius and π = relation between length of circumference and diameter = 3·1416 or approximately $\frac{22}{7}$.

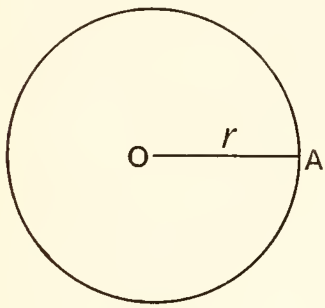

FIG. 16.

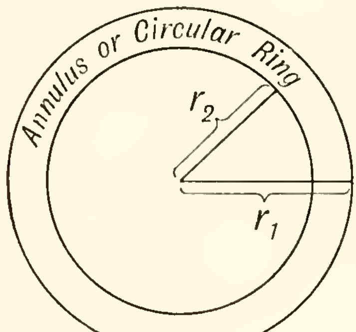

FIG. 17.

Area of Annulus, or Circular Ring (fig. 17).—In the diagram it is seen that the area of the circular ring is equal to the difference between the areas of the two concentric circles of radii r_1 and r_2 respectively.

$$\therefore \quad \text{Area of annulus} = \pi r_1^2 - \pi r_2^2$$
$$= \pi(r_1^2 - r_2^2)$$
$$= \pi(r_1 + r_2)(r_1 - r_2).$$

E.g. if r_1 = 10 in., and r_2 = 9 in., then

$$\text{Area of the annulus} = \pi(10+9)(10-9)$$
$$= 19\pi = 59{\cdot}7 \text{ sq. in.}$$

Area of a Sector of a Circle, OAMB (fig. 18). If θ be the value of the angle of the sector *in radians*, then

$$\frac{\text{Area of sector } (x)}{\text{Total area of circle } (\pi r^2)} = \frac{\text{Size of angle } \theta}{\text{Size of angle subtended by whole circumference } (2\pi).}$$

$$\therefore \qquad \text{Area of sector} = \frac{\theta \pi r^2}{2\pi} = \frac{\theta r^2}{2}.$$

If θ is in *degrees*, then area of sector $= \frac{\theta}{180} \cdot \frac{\pi r^2}{2} = \frac{\theta}{360}\pi r^2$.

Area of a Segment of a Circle, AMB (fig. 18).

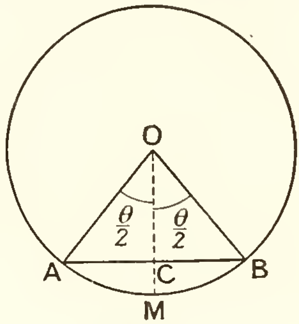

FIG. 18.

$$\begin{aligned}
\text{Area of AMB} &= \text{area of OAMB} - \text{area of triangle OAB} \\
&= \frac{\theta r^2}{2} - \tfrac{1}{2}\text{AB . OC} \\
&= \frac{\theta r^2}{2} - \text{AC . OC} \\
&= \frac{\theta r^2}{2} - r \sin \frac{\theta}{2} \,.\, r \cos \frac{\theta}{2} \\
&= \frac{\theta r^2}{2} - r^2 \sin \frac{\theta}{2} \,.\, \cos \frac{\theta}{2} \\
&= \frac{\theta r^2}{2} - \frac{r^2}{2} \sin \theta \text{ (p. 49)} \\
&= \frac{r^2}{2}(\theta - \sin \theta), \text{ when } \theta \text{ is in radians} \\
&= \frac{r^2}{2}\left(\frac{\theta^\circ \pi}{180} - \sin \theta\right), \text{ when } \theta \text{ is in degrees, etc.}
\end{aligned}$$

Cube (side a) (fig. 19).

Volume $= a^3$.
Surface $= 6a^2$.

$$\therefore \quad \frac{\text{Surface of cube}}{\text{Volume of cube}} = \frac{6a^2}{a^3} = \frac{6}{a}.$$

Corollary.—If side of cube is of unit length, then the extent of the surface is numerically six times the volume $\left(\text{since } \frac{6}{a} = \frac{6}{1} = 6\right)$.

If $a = 2$, then the extent of the surface is three times the volume $\left(\text{since } \frac{6}{a} = \frac{6}{2} = 3\right)$.

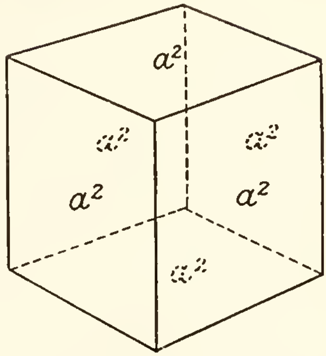

FIG. 19.

If $a = \frac{1}{2}$, then the extent of the surface is twelve times the volume $\left(\text{since } \frac{6}{a} = \frac{6}{0 \cdot 5} = 12\right)$.

Sphere (radius r, or diameter d).

Volume $= \frac{4\pi r^3}{3}$, or $\frac{\pi d^3}{6}$. *Surface* $= 4\pi r^2$, or πd^2 (see Chap. XVIII.).

$$\therefore \quad \text{Surface per unit volume} = \frac{4\pi r^2}{4\pi r^3/3} = \frac{3}{r}, \text{ or } \frac{\pi d^2}{\pi d^3/6} = \frac{6}{d}.$$

Hence, in the case of both the cube and the sphere, decrease in the length of the side—or diameter—leads to increase in the amount of surface area per unit volume, and *vice versa.*

Thus, a sphere 1 cm. in diameter has a volume of $\frac{3 \cdot 14}{6} = 0 \cdot 52$ c.c. and a surface area of 3·14 sq. cm., *i.e. 6 sq. cm. per c.c. volume.* If this sphere is broken up into, say, a billion ($= 10^{12}$) minute spheres of 1μ ($= 10^{-4}$ cm.) diameter, the aggregate volume would still be the same, viz. $10^{12}\pi(10^{-12})/6 = 0 \cdot 52$ c.c., but the total surface would be increased 10,000-fold to $10^{12}\pi(10^{-8}) = 3 \cdot 14 \times 10^4$ sq. cm., or to 6×10^4 sq. cm. ($= 6$ sq. metres) per c.c. volume. Colloidal particles of $0 \cdot 01\mu$ ($= 10^{-6}$ cm.) diameter have a surface of 600 sq. metres per c.c.!

Again, if two people, A and B, have the same amount of hæmoglobin per unit volume of blood, then the relative oxygen-carrying capacities of their red corpuscles are in the ratio of the aggregate surfaces of these corpuscles. If, for instance, A has 3·2 million red corpuscles, of average diameter $8 \cdot 3\mu$, per cubic mm. of blood, and B has, say, 4·9 million corpuscles, of average diameter $7 \cdot 5\mu$, per cubic mm., then assuming the corpuscles to be spherical the total corpuscular surfaces per cubic mm. of blood—or per unit amount of hæmoglobin—are in the ratio of

$$\frac{3 \cdot 2\pi(8 \cdot 3\mu)^2}{4 \cdot 9\pi(7 \cdot 5\mu)^2} = 0 \cdot 8.$$

Therefore A's blood has only 80 per cent. of the oxygen-carrying capacity of B's blood—notwithstanding the fact that it has exactly the same amount of hæmoglobin per unit volume of blood.

EXAMPLES.

(1) A person of normal sight can read print at such a distance that the letters subtend an angle of $5'$ at his eye. What is the height of the letters that he can read at a distance of 6 metres? Also, at what distance will a person whose vision is 6/36 be able to distinguish a man 180 cm. (= 6 ft.) tall?

The complete circumference (subtending 360°) of a circle of radius 6 metres $= 12\pi$ metres.

∴ Length of $5'$ arc (*i.e.* required height of the letters)

$$= \frac{5 \times 12\pi}{360 \times 60} \text{ metres} = 0{\cdot}875 \text{ cm.}$$

This is the principle upon which Snellen's type for testing distant vision is constructed. The letters which a normal person should read at 6 metres are 0·875 cm. (= 0·35 in.) in height. Those which he can see at 60 metres are necessarily 10 times as high, viz. 8·75 cm. (= 3·5 in.), and those he can see at 18 metres are $3 \times 0{\cdot}875 = 2{\cdot}625$ cm. (about 1 in.). If a person can see only at 6 metres what a normal person should see at 18 metres, or at 60 metres, respectively, then his vision is said to be 6/18, or 6/60, respectively.

A vision of 6/36 means ability to see only at 6 metres what a normal person should see at 36 metres, *i.e.* letters, etc., $6 \times 0{\cdot}875$ = about 5 cm. high. Hence such a person should be able to distinguish a man 180 cm. high at a distance 36 times (= 180/5) as far as 6 metres = 216 metres = 708 feet.

(2) A pendulum 3·5 feet long oscillates through an angle of 8° on either side of the vertical. Find (*a*) the length of the arc described, (*b*) the horizontal distance between the highest points, (*c*) the area of the sector of the circle described by the pendulum as it moves between the two highest points, (*d*) area of segment of circle between horizontal joining highest points and arc described by bob.

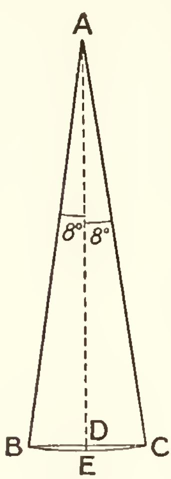

FIG. 20.

(*a*) The complete circumference (subtending 360°) of the circle of radius 3·5 feet or 42 in. is $2\pi \times 42 = 84\pi$ in.

$$\therefore \text{ Length of arc of } 16° = \frac{16 \times 84\pi}{360} \text{ in.} = 11{\cdot}73 \text{ in.}$$

$(\pi = \frac{22}{7})$.

(*b*) In the accompanying diagram (fig. 20) AB and AC represent the two extreme positions of the pendulum.

The line DC = AC sin DAC = 42 sin 8° = $42 \times 0{\cdot}1392$ in.

Also BD = DC.

∴ BC = $84 \times 0{\cdot}1392$ = 11·69 in.

(*c*) Area of sector ABEC $= \frac{\theta}{360}\pi r^2$ (where $\theta = 16°$, $\pi = \frac{22}{7}$ and $r = 42$ in.)

$= 246{\cdot}4$ sq. in.

(*d*) Area of segment BEC $= \frac{r^2}{2}\left(\frac{\theta\pi}{180} - \sin\theta\right)$

$$= \frac{42^2}{2}\left(\frac{16 \times 22}{7 \times 180} - \sin 16°\right)$$

$$= 882(0{\cdot}2794 - 0{\cdot}2756)$$

$$= 3{\cdot}4 \text{ sq. in.}$$

Prism (rectangular).—If a, b, c be the lengths of the sides of prism (fig. 21), then

$$\begin{aligned} \textit{Area} &= 2(ab+bc+ac). \\ \textit{Volume} &= abc. \\ \text{Diagonal BE} &= \sqrt{\text{EG}^2+\text{GB}^2} \\ &= \sqrt{\text{EG}^2+\text{GC}^2+\text{BC}^2} \\ &= \sqrt{c^2+a^2+b^2}. \end{aligned}$$

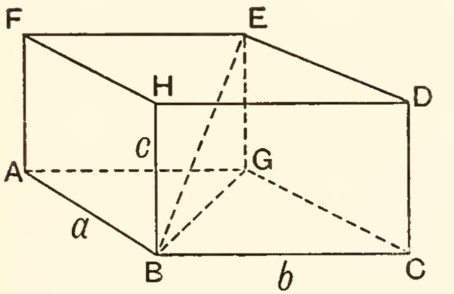

FIG. 21.

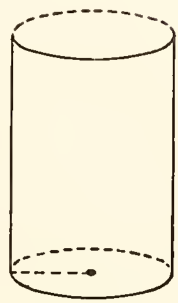
FIG. 22.

Cylinder (fig. 22).—

$$\begin{aligned} \textit{Area of curved surface} &= \text{circumference of base} \times \text{height}. \\ &= 2\pi rh. \\ \textit{Volume} &= \text{area of base} \times \text{height}. \\ &= \pi r^2 h. \end{aligned}$$

$$\begin{aligned} \textit{Total area} &= \text{area of curved surface} + \text{areas of two ends}. \\ &= 2\pi rh + 2\pi r^2. \\ &= 2\pi r(r+h). \end{aligned}$$

Cone (fig. 23).—*Volume* $= \frac{1}{3}\pi r^2 h$.

$$\textit{Curved surface} = \pi rl = \pi r\sqrt{h^2+r^2}.$$

$$\begin{aligned} \textit{Total surface} &= \text{area of curved surface} + \text{area of base}. \\ &= \pi rl + \pi r^2 = \pi r(l+r). \end{aligned}$$

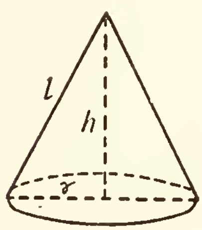

FIG. 23.

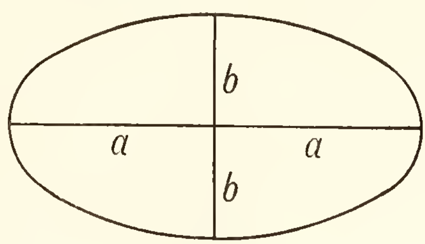

FIG. 24.

Ellipse (fig. 24).—If $2a$ and $2b$ denote the lengths of the major and minor axes respectively, then *circumference* $= \pi(a+b)$ approximately, and *area* $= \pi ab$.

EXAMPLES.

(1) Assuming metabolism to be proportional to amount of surface area, compare the rate of metabolism in man, whose surface area is about 260 square centimetres per kilo of body weight, with that of a cylindrical *bacillus coli*, whose size is $\mu \times 2\mu$, assuming its specific gravity to be 1, and $\mu = 10^{-4}$ cm.

$$\text{Volume of bacillus} = \pi r^2 h = \pi\left(\frac{\mu}{2}\right)^2 2\mu = \frac{\pi \times 10^{-12}}{2} \text{ c.c.}$$

$$\therefore \text{ Weight of bacillus} = \frac{\pi \times 10^{-12}}{2} \text{ gramme.}$$

$$\text{Surface of bacillus} = 2\pi r(r+h) = 2\pi\left(\frac{\mu}{2}\right)\left(\frac{\mu}{2}+2\mu\right) = \frac{5\pi \times 10^{-8}}{2} \text{ sq. cm.}$$

$$\therefore \text{ Surface per gramme of bacillus} = \frac{5\pi \times 10^{-8}}{\pi \times 10^{-12}} = 5 \times 10^4 \text{ sq. cm.}$$

and surface per kilogram of bacillus $= 5 \times 10^7$ sq. cm.

$$\therefore \quad \frac{\text{Surface per kilo of bacillus}}{\text{Surface per kilo of man}} = \frac{5 \times 10^7}{260} = \text{about } 2 \times 10^5.$$

Hence, metabolism in bacteria is about 200,000 times as great as in man. It is on account of this enormous rate of metabolism and absorption of food material that bacterial growth is so rapid—division taking place at the rate of about once in half an hour.

(2) If in fig. 21 AB (or GC) = 4 feet, BC (or AG) = 5 feet, and BH (or GE) = 2 feet, find the length of BE and its inclination to the base ABCG (or to line BG):

$$\text{BE} = \sqrt{4+16+25} = 3\sqrt{5} = 6{\cdot}708,$$

$$\sin \text{EBG} = \frac{\text{EG}}{\text{BE}} = \frac{2}{6{\cdot}708} = 0{\cdot}2982.$$

$$\therefore \quad \text{EBG} = 17^\circ\ 20'.$$

(3) Find the radius of a circle equal in area to that of an ellipse whose axes are 10 and 8 ft.

If r = radius of the circle, then its area $= \pi r^2$.

But area of ellipse $= \pi \times \dfrac{10}{2} \times \dfrac{8}{2} = 20\pi$ sq. ft.

$$\therefore \quad \pi r^2 = 20\pi$$
$$\therefore \quad r^2 = 20$$

whence
$$r = \sqrt{20} = 2\sqrt{5}$$
$$= 4{\cdot}472.$$

(4) It has been found that the average diameter of an adult's pulmonary air-cell = 0·2 mm., whilst that of an infant's air-cell (at birth) = 0·07 mm. Assuming that these air-cells are spherical, and that the total volume of the lungs = 1617 c.c. in the adult, and 67·7 c.c. in the new-born infant, find the total number of air-cells and their total surface in the adult and n the new-born infant.

Volume of single air-cell in adult $= \frac{4}{3}\pi(0{\cdot}1)^3$ cub. mm. (p. 55)
$= 0{\cdot}004$ cub. mm.

Volume of single air-cell in new-born $= \frac{4}{3}\pi(0{\cdot}035)^3$ cub. mm.
$= 0{\cdot}00018$ cub. mm.

$\therefore$ total number of air-cells in adult $= \dfrac{1617 \times 10^3}{0{\cdot}004} = 404 \times 10^6$.

And total number of air-cells in new-born $= \dfrac{67{\cdot}7 \times 10^3}{0{\cdot}00018} = 376 \times 10^6$,

i.e. **the number is approximately the same at birth as in the full-grown adult, viz. about 4×10^8.**

Surface of single air-cell in adult $= 4\pi(0{\cdot}1)^2 = 0{\cdot}125$ sq. mm.
Surface of single air-cell in new-born $= 4\pi(0{\cdot}035)^2 = 0{\cdot}0154$ sq. mm.
$\therefore$ Total surface of air-cells in adult $= 4 \times 10^8 \times 0{\cdot}125$ sq. mm.
$= 50$ sq. metres.
And total surface of air-cells in new-born $= 0{\cdot}0154 \times 4 \times 10^8$ sq. mm.
$= 6$ sq. metres.
$\therefore$ Total surface of air-cells in new-born is $\frac{1}{8}$ that in the adult.

Hence we see that whilst the volume of the infant's lungs is only about $\frac{1}{24}$ that in the adult, the **total surface of the alveoli is as much as $\frac{1}{8}$ that in the adult**—showing that the gaseous exchange is very active in young infants, *i.e.* about three times as active as in the adult. Moreover, as the area of the infant's skin surface is $\frac{1}{8}$ that in the adult, it is seen that the amount of gaseous interchange per unit of body surface is the same in the infant as in the adult. (See W. M. Feldman, "*Principles of Ante-Natal and Post-Natal Child Physiology,*" Longmans, 1920, and "*Principles of Ante-Natal and Post-Natal Child Hygiene,*" John Bale, Sons & Danielsson, 1927.)

(5) The following has been found to be the percentage composition of ordinary bacteria: water, 85 per cent.; solids, 15 per cent., of which 1 part in a thousand consists of sulphur.

Assuming that the weight of a molecule of any element $= M \times 8{\cdot}6 \times 10^{-22}$ mgm., where $M =$ molecular weight of the element, how many molecules of sulphur does a micrococcus of diameter $0{\cdot}15\mu$ ($\mu = \frac{1}{1000}$ mm.) contain?

Assuming the micrococcus to be spherical, its volume

$$= \tfrac{4}{3}\pi(0{\cdot}075\mu)^3$$
$$= 18 \times 10^{-13} \text{ cub. mm.}$$

Taking its sp. gr. $= 1$,
then its weight $= 18 \times 10^{-13}$ mgm.

Now, weight of sulphur molecule

$$= 32 \times 8{\cdot}6 \times 10^{-22} \text{ (since mol. wt. of S} = 32)$$
$$= 275 \times 10^{-22} \text{ mgm.}$$

But micrococcus contains $\frac{15}{100} \times \frac{1}{1000}$ part of sulphur
$= 15 \times 10^{-5}$ part of sulphur.

$\therefore$ Total weight of sulphur in micrococcus

$$= 15 \times 10^{-5} \times 18 \times 10^{-13} = 27 \times 10^{-17} \text{ mgm.}$$

But weight of one molecule of sulphur $= 275 \times 10^{-22}$ mgm.

$\therefore$ Number of molecules of sulphur in one micrococcus

$$= \frac{27 \times 10^{-17}}{275 \times 10^{-22}} = \text{about } 10{,}000.$$

(6) *Mode of Action of Renal Glomeruli.*—Brodie, by measuring the calibre of the tubules of the kidneys and the application of Poisseuille's law for

the passage of fluids along narrow tubes, finds that the pressure necessary to drive the fluid along the tubules is comparable to that existing in the glomerular capillaries. Hence he believes that the pulsation of the glomerulus does not account for the flow of urine in the tubules and, accordingly, attributes a secretory action to the glomerular surface.

The following are his calculations:—

Data.	Length. Cm.	Diameter. μ (*i.e.* $\frac{1}{1000}$ mm.).
(i) Proximal convoluted tubule . .	1·2	12
Loop of Henle—		
Descending limb	0·9	10
Ascending limb	0·9	9
Distal convoluted tubule . . .	0·2	18
Collecting tubule	2·2	16

(ii) Diuresis [*i.e.* rate at which urine was being discharged from one of the kidneys (containing 142,000 glomeruli and tubules) at the time of the experiment] = 1 c.c. per min., *i.e.* 1/142,000 c.c. per tubule per minute.

(iii) Poisseuille's law, which gives the pressure head of fluid that must have existed within *each* capsule in order to drive fluid out of the kidney, is

$p = \dfrac{8l\eta V}{\pi r^4}$ dynes per sq. cm. (see p. 254),

where

l = length of tube in cm.,

η = coefficient of viscosity of water at 35° C.
$= 719 \times 10^{-5}$,

V = flow in c.c. per second in each tubule $= \dfrac{1}{60} \cdot \dfrac{1}{142{,}000}$ c.c.

r = radius of tube in cm.

Hence

$$p = \frac{8 \times 719 \times 10^{-5}}{\pi} \cdot \frac{1}{60} \cdot \frac{1}{142000} \cdot 10^{16} \cdot \frac{l}{r^4} \text{ dynes per sq. cm. (where } r \text{ is expressed in } \mu)$$

$$= \frac{8 \times 719 \times 7}{22 \times 6 \times 142 \times 1333{\cdot}2} \cdot 10^{7} \cdot \frac{l}{r^4} \text{ mm. Hg.*}$$

$$= 1{\cdot}611 \times 10^{4}\frac{l}{r^4} \text{ mm. Hg.}$$

---

* A *dyne* is the c.g.s. unit of force, *i.e.* the force which acting upon a mass of one gramme for one second will impart to it a velocity of one centimetre per second; or, the force which acting continuously on a mass of one gramme (*e.g.* a column of water one centimetre high and one square centimetre cross-section) will impart to it an acceleration of 1 cm./sec.^{2}. As the acceleration due to gravity is 981 cm./sec.^{2}, it follows that 981 dynes per sq. cm. acting upward on a column of water 1 cm. high will counteract the pull upon the column of the force of gravity. In other words, the pressure of one centimetre of water is equivalent to 981 dynes per sq. cm. Hence the pressure of one mm. of water = 98·1 dynes per sq. cm. and the pressure of 1 mm. of mercury (whose sp. gr. is 13·59) is 98·1 × 13·59 = 1333·2 dynes per sq. cm. Therefore 1 dyne per sq. cm. = pressure of $\dfrac{1}{1333{\cdot}2}$ mm. Hg.

Consequently, for a flow of 1 c.c. per minute from *all* the tubules together

p, per centimetre of tubule, when	r is $4{\cdot}5\,\mu$	$= 39{\cdot}29$ mm. Hg.
	r is $5\,\mu$	$= 25{\cdot}78$,,
	r is $6\,\mu$	$= 12{\cdot}43$,,
	r is $8\,\mu$	$= 3{\cdot}93$,,
	r is $9\,\mu$	$= 2{\cdot}46$,,

Hence pressure head required for:

Proximal convoluted tubule	$= 1{\cdot}2 \times 12{\cdot}43 = 14{\cdot}916$	mm. Hg.
Loop of Henle—		
Descending limb	$= 0{\cdot}9 \times 25{\cdot}78 = 23{\cdot}202$	,,
Ascending limb	$= 0{\cdot}9 \times 39{\cdot}29 = 35{\cdot}361$	,,
Distal convoluted tubule	$= 0{\cdot}2 \times 2{\cdot}46 = 0{\cdot}492$	,,
Collecting tubule	$= 2{\cdot}2 \times 3{\cdot}93 = 8{\cdot}646$	,,
Total pressure head	$= 82{\cdot}617$	,,

As the mean aortic blood pressure was 120 mm. Hg., and the loss of pressure head between aorta and glomerular capillaries is about 35 mm. Hg., $\therefore$ blood pressure within the glomerular capillaries was probably about 85 mm. Hg.

In other words, practically the whole of the blood pressure is required to set up a pressure head in the fluid within the capsule sufficient to drive the secreted fluid down the tubules.

(*N.B.*—The bulk of opinion is against Brodie's conclusions. For criticism of the inference drawn from this calculation, see Cushny, "*The Secretion of Urine*," p. 57, Longmans, 1917.)

(7) The average diameter of a human capillary is $\frac{1}{100}$ mm.; the linear velocity of blood in it is $\frac{1}{2}$ mm. per second.

Find the volume of outflow from a capillary per second.

$$\begin{aligned} \text{Volume of outflow} &= \text{linear velocity} \times \text{cross-section.} \\ &= \tfrac{1}{2} \times \pi(\tfrac{1}{200})^2. \\ &= \tfrac{1}{2} \times 3{\cdot}14 \times \tfrac{1}{40{,}000} \\ &= 0{\cdot}00004 \text{ cub. mm. per second.} \end{aligned}$$

(8) Experiments on animals have shown that the circulation time is equal to 28 heart-beats. Assuming this to hold good for man, and also assuming the total volume of blood in the body to be 4000 c.c., find the number of capillaries in the human body, using the data of the last example, and assuming the pulse rate to be 72 per minute.

Since circulation time $= 28$ heart-beats,

and since 72 beats $= 1$ minute $= 60$ seconds,

$\therefore$ circulation time $= \frac{28}{72} \times 60 = 23{\cdot}3$ seconds.

But volume of capillary outflow $= 0{\cdot}00004$ cub. mm. per second.

$\therefore$ Total volume of outflow from one capillary in the circulation time $= 23{\cdot}3 \times 0{\cdot}00004 = 0{\cdot}000932$ cub. mm.

$\therefore$ If $n =$ number of capillaries in the body (filled with blood) we must have $0{\cdot}000932n =$ total volume of blood in body $= 4{,}000{,}000$ cub. mm.

$$\therefore \quad n = \frac{4 \times 10^6}{932 \times 10^{-6}} = 4{\cdot}3 \times 10^9.$$

(See W. M. Feldman, *Proc. Physiol. Soc.*, 1912.)

EXERCISES.

(1) The weight of a spherical shell is $\frac{7}{8}$ of what it would be if solid. Compare the inner and outer radii. If the inner radius be increased by one-half, find in what ratio the weight is reduced.

[*Answer.* If r_1 and r_2 are the inner and outer radii, respectively, $\frac{r_1^3}{r_2^3} = \frac{1}{8}$, $\therefore \frac{r_1}{r_2} = \frac{1}{2}$. If r_1 becomes 1·5, then weight of shell would be reduced from 7 to $8 - (1{\cdot}5)^3$, *i.e.* from 7 to 4·625 or in proportion of $\frac{7}{4{\cdot}625} = 56 : 37$.]

(2) The radius of each particle of cholesterol in a sol containing 0·0005 gramme of that substance per c.c., is 10^{-6} cm. Find the total surface area of these particles. (Assume sp. gr. = 1.)

[*Answer.* Volume or mass of each particle $= \frac{4}{3}\pi \times 10^{-18}$. Number of particles in 1 c.c. $= \dfrac{5 \times 10^{-4}}{\frac{4\pi}{3} 10^{-18}} = \dfrac{15}{4\pi} \,.\, 10^{14}$. $\therefore$ Total surface area $= 4\pi \,.\, 10^{-12} \times \dfrac{15}{4\pi} \,.\, 10^{14} = 1500$ sq. cm.]

CHAPTER VI.

SERIES.

A succession of quantities, each of which is formed in accordance with some fixed rule or principle, is called a *series*. Each of the successive quantities in a series is called a *term*, and the fixed rule in accordance with which each term is formed is called the *law of the series.*

There are very many different kinds of series, with some of which we shall have to deal, *e.g.* the series e (p. 74), the exponential series (p. 80), the logarithmic series (p. 81), the various trigonometrical series (p. 229), etc. The best known types are the arithmetical and geometrical progressions.

An **arithmetical progression** (A.P.) is a series in which each term differs from its immediate predecessor by a constant quantity called the *common difference* (C.D.). Thus

$$1, 3, 5, 7, 9 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$6, 3, 0, -3, -6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

are arithmetical progressions, the C.D. in the first being 2, $(3-1 = 5-3$, etc., $= 2)$, and that in the second -3, $(3-6 = 0-3$, etc., $= -3)$.

The general form of an A.P. is

$$a, a+d, a+2d, a+3d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

where a is the first term, and d is the common difference (it may be +ve or −ve, integral or fractional). The nth term obviously is $a+(n-1)d$.

Whenever any quantity grows in such a way that its increase (or decrease) in value during any equal intervals of time is always a constant proportion of its original value, then the successive values attained by it at the ends of these intervals of time form the terms of an arithmetical progression.

The best example of such a form of growth is money lent at a fixed rate of *simple interest.*

Thus, if £100 be invested at 10 per cent. simple interest per annum, then each year the capital increases by $\frac{1}{10}$ of its original

value, viz. £10, and the amounts to which the capital has grown at the beginning of each successive year are the terms of the A.P.

100, 110, 120, 130 . . . .

Hence we can say that *the law of an arithmetical progression is analogous to the law of simple interest.*

The Arithmetic Mean (*A.M.*) of two numbers is half the sum of these numbers. Thus the A.M. of 5 and 9 is $\frac{5+9}{2} = 7$. The A.M. of a series of n different numbers is $1/n$ of their sum. Thus the A.M. of 3, 5, 10 and 12 $= \frac{3+5+10+12}{4} = 7{\cdot}5$. In the case of an arithmetical progression, the A.M. of any number of terms of the series is equal to half the sum of the first and last terms. Thus $\frac{3+5+7+9}{4} = \frac{3+9}{2} = 6$.

A **geometrical progression** (G.P.) is a series in which each term bears a constant ratio—called the *common ratio* (C.R.) to its immediate predecessor. Thus

$3, 6, 12, 24, 48$ . . . .
$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$ . . . .

are geometrical progressions, the C.R. in the first being 2, ($\frac{6}{3} = \frac{12}{6}$, etc., $= 2$), and that in the second being $\frac{1}{3}$, ($\frac{1}{3}/1 = \frac{1}{9}/\frac{1}{3}$, etc., $= \frac{1}{3}$).

The general form of a G.P. is

a, ar, ar^2, ar^3 . . . .

where a is the first term, and r is the common ratio (integral or fractional, + ve or − ve). The nth term is obviously ar^{n-1}.

Whenever any quantity grows in such a way that its increase (or decrease) in value during any equal intervals of time is proportional not to its original value, but to its value at the beginning of the interval, then the successive values attained by it at the ends of these intervals of time form the terms of a geometrical progression.

The best example of such a form of growth is money invested at a fixed rate of *compound* interest.

Thus, if £100 be invested at 10 per cent. compound interest per annum, then the interest during the first year $= \frac{1}{10} \times 100$ $=$ £10. This, being added to the capital, makes the new capital at the end of the first or beginning of the second year

$$= 100 + \tfrac{1}{10} \,.\, 100 = £100(1 + \tfrac{1}{10}).$$

$\therefore$ The interest during the second year $= \frac{1}{10} \times 100(1+\frac{1}{10})$, which, when added to the increased capital, makes the new capital at the end of the second or beginning of the third year

$$= 100(1+\tfrac{1}{10}) + \tfrac{1}{10} \times 100(1+\tfrac{1}{10})$$
$$= 100(1+\tfrac{1}{10})(1+\tfrac{1}{10}) = £100(1+\tfrac{1}{10})^2.$$

$\therefore$ The interest during the third year $= \frac{1}{10} \times 100(1+\frac{1}{10})^2$, which, when added to the further increased capital, makes the new capital at the end of the third or beginning of the fourth year

$$= 100(1+\tfrac{1}{10})^2 + \tfrac{1}{10} \times 100(1+\tfrac{1}{10})^2$$
$$= 100(1+\tfrac{1}{10})^2(1+\tfrac{1}{10}) = £100(1+\tfrac{1}{10})^3,$$

and so on.

So that the amounts to which the capital has grown at the beginning of each successive year are the terms of the G.P.

$$100,\ 100(1+\tfrac{1}{10}),\ 100(1+\tfrac{1}{10})^2,\ 100(1+\tfrac{1}{10})^3 \ldots$$

or $$100,\ 100(1{\cdot}1),\ 100(1{\cdot}1)^2,\ 100(1{\cdot}1)^3.\ \ldots$$

Hence we can say that *the law of a geometrical progression is analogous to the law of compound interest.*

We shall return to the compound interest law presently (see p. 87).

The Geometric Mean (*G.M.*) of two numbers is the square root of their product. Thus the G.M. of 5 and 45 is $\sqrt{5 \times 45} = 15$. The G.M. of a series of n different numbers is the nth root of their product. Thus the G.M. of 2, 3, 5 and 10 $= \sqrt[4]{2 \times 3 \times 5 \times 10}$. In the case of a geometrical progression, the G.M. of any number of terms of the series is equal to the square root of the product of the first and last terms. Thus the G.M. of 2, 4, 8, 16 $= \sqrt[4]{2 \times 4 \times 8 \times 16} = \sqrt{2 \times 16} = 4\sqrt{2}$.

Note.—The G.M. may be expressed logarithmically as a Sum. Thus, if we call the G.M. of 2, 4, 8, 16 . . . to n terms g, then

$$g = \sqrt[n]{2 \times 4 \times 8 \times 16 \ldots}$$

$$\therefore \quad \log g = \frac{1}{n}(\log 2 + \log 4 + \log 8 + \log 16 \ldots).$$

$$= \frac{\log 2}{n}(1+2+3+4+\ldots+n)$$

$$= \frac{\log 2}{n}\frac{(1+n)n}{2} = \frac{(1+n)\log 2}{2}$$ (see sum of first n natural numbers, p. 69.)

Harmonic Progressions are not important enough for our purpose to be treated here.

The Binomial Theorem.—A most important and interesting series is obtained by raising a binomial expression (*i.e.* an expression containing two terms, like $(a+b)$, etc.) to any power n. Newton's *Binomial Theorem* states that for *any* value of a, b and n,

$$\begin{aligned}(a+b)^n = a^n &+ \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 \\ &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^3b^{n-3} + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-2} \\ &+ \frac{n}{1}ab^{n-1} + b^n.\end{aligned}$$

The right-hand side of this identity is called the *expansion* of $(a+b)^n$.

There is no need to give here a very rigid proof of this theorem. It will be sufficient for our purpose if we verify the theorem for several different values of n.

Thus we know from actual multiplication that

$(a+b)^2 = a^2 + 2ab + b^2$, which is the same as

$$a^2 + \frac{2}{1}a^{2-1}b + \frac{2(2-1)}{1 \cdot 2}a^{2-2}b^2 + \left\{\frac{2(2-1)(2-2)}{1 \cdot 2 \cdot 3}a^{2-3}b^3 + \quad . \quad . \quad . \right\}$$

since $$\frac{2}{1}a^{2-1}b = 2ab$$

$$\frac{2(2-1)}{1 \cdot 2}a^{2-2}b^2 = \frac{2 \cdot 1}{1 \cdot 2}a^0b^2 = b^2,$$

and each of the subsequent terms contained in the brackets { } is equal to 0, because it contains the factor $(2-2)$ which equals 0.

Similarly, actual multiplication gives

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \text{ which is the same as}$$

$$a^3 + \frac{3}{1}a^{3-1}b + \frac{3(3-1)}{1 \cdot 2}a^{3-2}b^2 + \frac{3(3-1)(3-2)}{1 \cdot 2 \cdot 3}a^{3-3}b^3.$$

All the subsequent terms, containing as they do $(3-3)$ as a factor, vanish.

And so on, for any value of n.

The Factorial Notation.—It is customary to denote the product of the 1st n consecutive integers

$$1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,(n-2)(n-1)n \text{ as } \underline{|n} \text{ or } n!$$

which is read *factorial n.*

Thus

$$\underline{|1} \text{ or } 1! = 1$$

$$\underline{|2} \text{ or } 2! = 1\,.\,2 = 2$$

$$\underline{|3} \text{ or } 3! = 1\,.\,2\,.\,3 = 6$$

$$\underline{|4} \text{ or } 4! = 1\,.\,2\,.\,3\,.\,4 = 24,$$

and so on.

Hence, the binomial theorem may be written as

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2$$

$$+\ .\ .\ .\ + \frac{n(n-1)}{2!}a^2b^{n-2} + \frac{n}{1!}ab^{n-1} + b^n.$$

It will be noted that the coefficient of the rth term is

$$\frac{n(n-1)(n-2)\ .\ .\ .\ (n-r+2)}{1\,.\,2\,.\,3\ .\ .\ .\ (r-1)}.$$

Thus when $r = 3$ it becomes $\dfrac{n(n-1)}{1\,.\,2}$, and so on.

Similarly,

$$(a-b)^n = a^n + na^{n-1}(-b) + \frac{n(n-1)}{1\,.\,2}a^{n-2}(-b)^2$$

$$+ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}(-b)^3 + \ .\ .\ .$$

$$= a^n - na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{1\,.\,2} - \frac{n(n-1)(n-2)a^{n-3}b^3}{1\,.\,2\,.\,3} + \ldots$$

the $+$ and $-$ signs alternating continually, provided n is a positive integer.

The student will note that the coefficients of the first, second, third, etc., terms from the beginning are respectively equal to those of the first, second, third, etc., terms from the end.

EXAMPLES.

(1) $\sqrt{10} = \sqrt{9+1} = \sqrt{9(1+\frac{1}{9})} = 3(1+\frac{1}{9})^{\frac{1}{2}}$

$$= 3\left\{1 + \frac{1/2}{1}\cdot\frac{1}{9} + \frac{1/2(1/2-1)}{1\,.\,2}\cdot\frac{1}{9^2}\right.$$

$$\left. + \frac{1/2(1/2-1)(1/2-2)}{1\,.\,2\,.\,3}\cdot\frac{1}{9^3} + \ .\ .\ .\right\}$$

$$= 3(1 + \tfrac{1}{2}\cdot\tfrac{1}{9} - \tfrac{1}{8}\cdot\tfrac{1}{81} + \tfrac{1}{16}\cdot\tfrac{1}{729} - \ .\ .\ .)$$

$$= 3{\cdot}1623\ .\ .\ .$$

(2) $$\sqrt{0\cdot8} = (1-0\cdot2)^{\frac{1}{2}} = 1 - \frac{1/2}{1}\,.\,(0\cdot2) + \frac{1/2(1/2-1)}{1\,.\,2}\,.\,(0\cdot2)^2 - \frac{\frac{1}{2}(1/2-1)(1/2-2)}{1\,.\,2\,.\,3}\,.\,(0\cdot2)^3 + \ldots$$
$$= 1 - \frac{0\cdot2}{2} - \frac{(0\cdot2)^2}{8} - \frac{(0\cdot2)^3}{16} - \ldots$$
$$= 0\cdot8944 \ldots$$

(3) $\sqrt[4]{1\cdot006} = (1+0\cdot006)^{\frac{1}{4}} = 1 + \frac{1}{4} \times 0\cdot006 - \ldots$
$= 1\cdot0015$, correct to four places of decimals.

(4) $(1\cdot006)^4 = 1 + 4 \times 0\cdot006 + \frac{4\,.\,3}{1\,.\,2}(0\cdot006)^2 + \ldots = 1\cdot024$, correct to three decimal places.

From the last two examples we learn that if a is very small compared with unity, so that terms containing a^2 and higher powers of a may be neglected, then

$$(1+a)^n = 1 + na.$$
$$(1+a)^{\frac{1}{n}} = 1 + \frac{a}{n}, \text{ i.e. } \sqrt[n]{1+a} = 1 + \frac{a}{n}.$$

Also $$(1+a)^{-n} = 1 - na, \text{ i.e. } \frac{1}{(1+a)^n} = 1 - na.$$

and $$(1+a)^{-\frac{1}{n}} = 1 - \frac{a}{n}, \text{ i.e. } \sqrt[n]{\frac{1}{1+a}} = 1 - \frac{a}{n}.$$

Similarly, if a and b be very small compared with unity, then, since $$(1 \pm a)(1 \pm b) = 1 \pm a \pm b \pm ab,$$
we may for purposes of approximation write

$$(1 \pm a)(1 \pm b) = 1 \pm a \pm b,$$

since ab, being the product of two very small quantities, becomes negligibly small.

Thus $$(1\cdot0003) \times (1\cdot0006) = 1 + 0\cdot0003 + 0\cdot0006 = 1\cdot0009.$$

Also $$(1 \pm a)^m(1 \pm b)^n = (1 \pm ma)(1 \pm nb) = 1 \pm ma \pm nb.$$

E.g. if t is small $\sqrt{1 + \frac{t}{273}} = \left(1 + \frac{t}{273}\right)^{\frac{1}{2}} = 1 + \frac{t}{546}.$

EXERCISES.

(1) Find $\sqrt{2}$ by means of the binomial theorem. [*Answer*, 1·414.]

(2) Prove that $\sqrt[3]{26} = 2\cdot9625$.

There are many other types of series with which we shall have to deal in this book, *e.g.* the exponential, the logarithmic (pp. 80 and 81), the various trigonometrical series (p. 229), and the various series that result from the expansion of binomial expressions. The one which occurs with the utmost frequency in the higher mathematics is:

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$$

or $$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

This, as we shall see later (p. 77), is a series, the successive terms of which represent the successive terms in the expansion $\left(1 + \frac{1}{n}\right)^n$, when n is infinitely large, and is called the series e.

Summation of Series.—Once the law of a series is discovered, it is as a rule possible to sum the series.

(a) *Arithmetical Progression.*

Let $S = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$, where l represents the nth term.

Writing the series in the reverse order,

$$S = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a.$$

By addition,

$$\begin{aligned} 2S &= (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) \\ &\qquad + (a + l), \text{ repeated } n \text{ times} \\ &= n(a + l). \end{aligned}$$

$$\therefore \quad S = \frac{n}{2}(a + l) = \frac{n}{2}\{a + [a + (n - 1)d]\} = \frac{n}{2}\{2a + (n - 1)d\}.$$

Thus the sum of the first n natural numbers, $1 + 2 + 3 + \ldots + n$ is $\frac{n}{2}(1 + n)$.

EXAMPLES.

(1) The sum of $1 + 8 + 15 + \ldots$ to 40 terms

$$= \tfrac{40}{2}\{2 + 39 \times 7\} = 20 \times 275 = 5500.$$

(2) How many strokes does a clock make in 24 hours, if it strikes 1 for the half-hours?

The number of strokes in 12 hours $= \frac{12}{2}\{1 + 12\} = 6 \times 13 = 78$, for the hours, together with 12 for the half-hours $= 90$ altogether.

Therefore the total number of strokes in the 24 hours $= 90 \times 2 = 180$.

(b) *Geometrical Progression.*

Let $S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$ where ar^{n-1} is the nth term.

$\therefore$ $Sr = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$.

By subtraction we get

$Sr - S = ar^n - a$ (all the intermediate terms cancelling themselves out),

i.e.
$$S(r-1) = a(r^n - 1)$$
$$\therefore \quad S = \frac{a(r^n-1)}{r-1}.$$

If $r < 1$, so that r^n is a fraction less than 1, then the sum is written as
$$S = \frac{a(1-r^n)}{1-r}.$$

Corollary.—Since when $r < 1$, $r^\infty = 0$ (see p. 7),

$\therefore$ When $r < 1$, the sum to infinity $= \dfrac{a}{1-r}$.

EXAMPLES.

(1) The sum to 10 terms of $1 + 2 + 4 + 8 + \ldots$ is $\dfrac{2^{10}-1}{2-1} = 1023$.

(2) The sum to infinity of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is $\dfrac{1}{1-\frac{1}{2}} = 2$.

(3) An elastic ball let fall on a table rebounds to a height which is e^2 times the height from which it falls (e being the coefficient of elasticity, which is less than 1). Find the space described by the ball if it is let fall originally from a height h and left to itself.

Distance of 1st fall $= h$.
Return distance of 1st rebound $= 2e^2h$,
,, ,, 2nd ,, $= 2e^4h$,
,, ,, 3rd ,, $= 2e^6h$,

and so on.

$\therefore$ Total distance covered $= h + 2e^2h(1 + e^2 + e^4 + \ldots$ to infinity)
$$= h + 2e^2h\left(\frac{1}{1-e^2}\right)$$
$$= \frac{h(1+2e^2)}{1-e^2}.$$

(4) Prove that $\sqrt{3\sqrt{3\sqrt{3 \ldots}}}$ to infinity $= 3$.

The expression $= 3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \ldots$ to infinity $= x$, say.

$\therefore$ $\log x = \log 3(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ to infinity)

$= \log 3$.

$\therefore$ Value of expression $= 3$. (*Cf.* Example (2), p. 22.)

(e) *The Sums of the Squares and of the Cubes of the First n Natural Numbers.*

(i) Let $S = 1^2 + 2^2 + 3^2 + \ldots + n^2$

$$\begin{aligned}
\text{Since } n^3 - (n-1)^3 &= 3n^2 - 3n + 1 \\
\text{and } (n-1)^3 - (n-2)^3 &= 3(n-1)^2 - 3(n-1) + 1 \\
&\;\cdots \\
2^3 - 1^3 &= 3 \times 2^2 - 3 \times 2 + 1 \\
1^3 - 0^3 &= 3 \times 1^2 - 3 \times 1 + 1
\end{aligned}$$

Adding $n^3 = 3S - \frac{3n}{2}(n+1) + n$ (the sum of $1 + 2 + 3 + \ldots + n$ being $\frac{n}{2}(n+1)$).

$$\therefore \quad S = \frac{1}{3}\left\{n^3 - n + \frac{3n}{2}(n+1)\right\} = \frac{n(n+1)(2n+1)}{6}.$$

(ii) In a similar manner it is easy to prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \{n(n+1)/2\}^2.$$

The sum of a series is sometimes denoted by writing the Greek letter Σ in front of the general term. Thus $\Sigma(1 + r^2)$ stands for, and is read as, "the sum of such terms as $(1 + r^2)$." By placing small letters after Σ, we indicate how many terms are to be taken, thus

$$\Sigma_{r=1}^{r=50}(1 + r^2)$$

denotes the sum of the terms obtained from $(1 + r^2)$ by giving r the values from 1 to 50 in succession.

$$\begin{aligned}
\therefore \quad \Sigma_{r=1}^{r=50} &= (1 + 1^2) + (1 + 2^2) + (1 + 3^2) + (1 + 4^2) + (1 + 5^2) \\
&\qquad + \ldots + (1 + 50^2) \\
&= 2 + 5 + 10 + 17 + 26 + \ldots + 2501.
\end{aligned}$$

When the successive terms differ from one another by infinitesimally small quantities, then the symbol $\int$ (which is a long S) is substituted for Σ. We shall return to this in our section on the Integral Calculus.

Finite and Infinite Series.—If a series terminates at some assigned term, say, the nth term—where n is a finite number, like 100, 200, 1000, and so on, then it is called a finite series, *e.g.*

1, 3, 5, 7, 9 . . . 29.

If, however, the number of terms is unlimited, it is called an infinite series, *e.g.*

1, 3, 5, 7, 9 . . . 27, 29 . . . to infinity.

Now whilst the sum of an infinite series like this will necessarily be infinitely great—increasing as it does with each term—there are some infinite series whose sum may have a **finite** value. A very good example of such a series is afforded by a recurring decimal. Thus, if we convert $\frac{1}{3}$ into a decimal fraction we obtain:

$\frac{1}{3} = 0{\cdot}33333 \ldots$ the 3's being continued for ever,

i.e. $\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$ to infinity.

The same is the case with $\frac{1}{9}$, for

$\frac{1}{9} = 0{\cdot}11111 \ldots$ the 1's being continued for ever,

$= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$ to infinity.

Each of these infinite series has a finite value, which means that if we continually increase the number of terms in each of these series the sum will get nearer and nearer to $\frac{1}{3}$ or $\frac{1}{9}$ respectively. Thus the difference between

$$\frac{1}{9} \text{ and } 0{\cdot}1111 \text{ is } \frac{1}{9} - \frac{1111}{10,000} = \frac{1}{90,000};$$

the difference between

$$\frac{1}{9} \text{ and } 0{\cdot}1111111111 \text{ is } \frac{1}{9} - \frac{1111111111}{10,000,000,000} = \frac{1}{9 \,.\, 10^{10}},$$

and so on.

Now, $\frac{1}{9 \,.\, 10^{10}}$ is such a small fraction that if we were to take this fraction of the velocity of light, which is 186,000 miles per second, it would amount only to about $\frac{1}{10}$ inch per second! Now, if this is the case when we take 10 terms of the series, imagine how much smaller the difference would be if we were to take 20, or 100, or 1000, or 1,000,000 terms. Indeed, it is obvious that the greater the number of terms in this series, the nearer and nearer its sum approaches to the value $\frac{1}{9}$, and if we continue the series to an infinite number of terms its sum *ultimately* becomes the value $\frac{1}{9}$. Now, such an **ultimate** value of the series—a value which the sum of the series never actually reaches, but approaches closer and closer, and to which it may get as close as ever we please by continuing the series long enough—is called the *limit* or *limiting value* of the series.

The symbolical way of writing this is $\mathrm{Lt}_{n\to\infty}\, 0{\cdot}111 \ldots = \frac{1}{9}$, which is read as follows: "The limit of $0{\cdot}111 \ldots$ to n terms when n is made infinitely great is equal to $\frac{1}{9}$." (The symbol ∞ stands for infinity.)

Similarly, if we take any fraction $\frac{1}{n}$, we can make it as small as we please by making n sufficiently large; thus if $n = 1000$, $\frac{1}{n} = \frac{1}{1000}$; if $n = 1{,}000{,}000$, $\frac{1}{n} = \frac{1}{1{,}000{,}000}$, and so on; so that by taking n very very large, $\frac{1}{n}$ becomes very very small, and when n is made infinitely large, *i.e.* $n = \infty$, then $\frac{1}{n}$ becomes $\frac{1}{\infty}$, *i.e.* infinitely small, *i.e.* 0. Hence, $\mathrm{Lt}_{n\to\infty}\frac{1}{n} = 0$, which is read as follows: "The limit of $\frac{1}{n}$ when n is infinitely large is zero."

Similarly, since by actual division we find that

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots \text{ to infinity,}$$

we say that $\mathrm{Lt}_{n\to\infty}(1 + x + x^2 + \ldots) = \frac{1}{1-x}$.

Also $\frac{1}{1+x} = 1 - x + x^2 - \ldots$ to infinity,

$$\therefore \quad \mathrm{Lt}_{n\to\infty}(1 - x + x^2 - \ldots) = \frac{1}{1+x}.$$

Convergency and Divergency of Series.—When an infinite series is of such a nature that its sum to any number of terms cannot numerically exceed some finite quantity—called the limit—however large the number of terms, then such a series is said to be *convergent*.

Thus, the sum of the series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{n-1}} = \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} \text{ (see p. 70).}$$

Now $\frac{1}{3^n}$ becomes smaller and smaller as n is made larger and larger, and when $n = \infty$, $\frac{1}{3^n} = 0$.

$$\therefore \quad \frac{1-\frac{1}{3^n}}{1-\frac{1}{3}} = \frac{1-0}{\frac{2}{3}} = \frac{1}{2/3} = \frac{3}{2}.$$

Hence the series $1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3} \ldots$ is convergent.

Every convergent series can be summed to any degree of accuracy desired.

When an infinite series is of such a nature that its sum to n terms can be made greater than any finite quantity by taking n large enough, then such a series is said to be *divergent*.

E.g. the sum of the series

$$1+3+3^2+3^3+ \ldots \text{ to } n \text{ terms} = \frac{3^n-1}{3-1},$$

and when $n = \infty$, 3^n, and hence also $3^n - 1 = \infty$,

$$\therefore \quad \frac{3^n-1}{3-1} = \infty.$$

Hence the series $1, 3, 3^2, 3^3 \ldots$ is divergent.

Obviously, a divergent series cannot be summed.

From what we have just said it follows, therefore, that the series

$$1+x+x^2+x^3+ \ldots$$

is convergent or divergent, according as $x < 1$ or > 1.

Thus, the sum of the series to n terms is

$$1+x+x^2+x^3+ \ldots +x^{n-1} = \frac{x^n-1}{x-1}.$$

If $x < 1$ and $n = \infty$, $x^n = 0$, and sum $= \dfrac{1}{1-x}$,

but if $x > 1$ and $n = \infty$, $x^n = \infty$, and sum $= \infty$.

The Series "e."—The compound interest law, with which we dealt on pp. 64, 65, leads us to another series which is the most important series in the higher mathematics, as we shall see presently (p. 80 *et seq.*).

The sum of this series, which, like π (*i.e.* the relation between the circumference of a circle and its diameter), is an incommensurable number—although, as we shall see, it can be calculated to as many places of decimals as we wish—is called **e**,

after Euler, the discoverer of the series. It is, as we have already seen on p. 69,

$$1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\ .\ .\ .$$

and its value to 5 decimal places is 2·71828.

If money, say, £100, is invested at simple interest at the rate, say, of 10 per cent. per annum, then since as we have seen the amounts to which the capital has grown at the ends of the first, second, third, etc., years are £110, £120, £130, etc., therefore in ten years the capital would become £200, *i.e.* the capital would double itself. And no matter whether the interest is collected in yearly instalments of £$\frac{100}{10}$, or in monthly instalments of £$\frac{100}{120}$, or in daily instalments of £$\frac{100}{3650}$, etc., the capital would still double itself in the same period of ten years. Indeed, if we were to divide the year into n intervals and collect £$\frac{10}{n}$ at the end of every interval, the amount collected during each year would still be £10 $\left(i.e.\ £\frac{10}{n}\times n\right)$, and the capital would still double itself in ten years. But in the case of compound interest it is different. We have seen that if the interest, say, 10 per cent., is collected **and added to the capital every year,** then the amounts to which the capital has grown at the ends of the first, second, third, etc., years are

$$£100(1+\tfrac{1}{10}),\quad £100(1+\tfrac{1}{10})^2,\quad £100(1+\tfrac{1}{10})^3,\text{ etc. (p. 65),}$$

so that in ten years the capital would become

$$£100(1+\tfrac{1}{10})^{10} = £100(1\cdot1)^{10}$$
$$= £100\times 2\cdot59375;\ [10\log 1\cdot1 = 0\cdot41393 = \log 2\cdot59375].$$

If, however, instead of collecting the interest yearly, we were to collect it monthly and add it to the capital each time, then the interest during the first month $= £\frac{1}{120}\times 100$.

This, added to the capital, makes the new capital at the end of the first month or beginning of second month

$$= 100+\tfrac{1}{120}\times 100 = £100(1+\tfrac{1}{120}).$$

$\therefore$ Interest during second month $= £\frac{1}{120}\times 100(1+\frac{1}{120})$, which, when added to the increased capital, makes the new capital at the end of the second or beginning of the third month

$$= 100(1+\tfrac{1}{120})+\tfrac{1}{120}\times 100(1+\tfrac{1}{120})$$
$$= 100(1+\tfrac{1}{120})(1+\tfrac{1}{120})$$
$$= £100(1+\tfrac{1}{120})^2.$$

Similarly, at the end of the third month the capital has grown to

$$£100(1+\tfrac{1}{120})^3$$

and so on.

So that at the end of ten years, or 120 months, when at simple interest the capital would have doubled itself, the capital would, at compound interest reckoned monthly, have grown to

$$£100(1+\tfrac{1}{120})^{120} = £100(\tfrac{121}{120})^{120} = £100 \times 2{\cdot}707$$

[since $120 \log \frac{121}{120} = 120(\log 121 - \log 120) = 0{\cdot}43248 = \log 2{\cdot}707$].

Supposing, however, we collected the interest and added it to the capital, not monthly, but daily.

Then, the interest during the first day $= £\frac{1}{3650} \times 100$.

This, added to the capital, makes the new capital at the end of the first or beginning of the second day $100 + \frac{1}{3650} \times 100 = £100(1+\frac{1}{3650})$.

$\therefore$ Interest during second day $= £\frac{1}{3650} \times 100(1+\frac{1}{3650})$,

which, when added to increased capital, makes the new capital at the end of the second day

$$= 100(1+\tfrac{1}{3650}) + \tfrac{1}{3650} \times 100(1+\tfrac{1}{3650}) = £100(1+\tfrac{1}{3650})^2$$

and so on.

So that at the end of 10 years, or 3650 days, when at simple interest the capital would have doubled itself, the capital would, at compound interest reckoned daily, have grown to

$$£100(1+\tfrac{1}{3650})^{3650} = £100(\tfrac{3651}{3650})^{3650} = £100 \times 2{\cdot}718$$

[$3650 \log \frac{3651}{3650} = 3650(\log 3651 - \log 3650) = 0{\cdot}43435 = \log 2{\cdot}718$].

If we go still further and calculate the interest every hour, then in 10 years, *i.e.* 87,600 hours, the £100 would become

$$£100(\tfrac{87601}{87600})^{87600} = £100 \times 2{\cdot}71828.$$

Finally, it will be seen that **since the capital keeps on growing every minute, second, and, indeed, every instant,** the ultimate value of the £100 at the end of 10 years will be $£100\left(1+\frac{1}{n}\right)^n$, where n (the number of intervals) is infinitely great.

Now, expanding $\left(1+\frac{1}{n}\right)^n$ by the binomial theorem (p. 66), we get

$$\left(1+\frac{1}{n}\right)^n = 1+\frac{n}{1}\cdot\frac{1}{n}+\frac{n(n-1)}{1\,.\,2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cdot\frac{1}{n^3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{1}{n^4}+\ .\ .\ .$$
$$= 1+\frac{1}{1}+\frac{n^2\left(1-\frac{1}{n}\right)}{1\,.\,2}\cdot\frac{1}{n^2}+\frac{n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\,.\,2\,.\,3}\cdot\frac{1}{n^3}$$
$$+\frac{n^4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{1}{n^4}+\ .\ .\ .$$

But when $n = \infty$, $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$, etc. $= 0$.

$$\therefore\ \mathrm{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\ .\ .\ .$$

which, when worked out to the first six decimal places, gives the result 2·718282 as follows:—

	1	=	1·000000 00
divided by	1	=	1·000000 00
,,	2	=	0·500000 00
,,	3	=	0·166666 66
,,	4	=	0·041666 66
,,	5	=	0·008333 33
,,	6	=	0·001388 88
,,	7	=	0·000198 41
,,	8	=	0·000024 80
,,	9	=	0·000002 76
,,	10	=	0·000000 27
			2·718282 . . .

This series $1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\ .\ .\ .$ *is called the series* e, which, as has been said, is the most important series in the higher mathematics. We can say, therefore, that *the series* e *is the expansion of* $\left(1+\frac{1}{n}\right)^n$ *when* n *is made infinitely large.*

The Meaning of "e."—From the method by which we have derived e $\left[i.e.\ \mathrm{e} = \mathrm{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]$ we see that e represents the

amount which a unit quantity, increasing **continuously** at a certain rate in accordance with the law of *compound* interest (*i.e.* at a rate which is a constant proportion of its magnitude at every instant), will reach after that period of time at which the same quantity would double itself if it grew at the same rate in accordance with the law of *simple* interest (*i.e.* at a rate which is proportional to its original value).

Thus, if two equal capitals be put out *at the same rate of interest* at the same time, one at simple and the other at compound interest, then at the time at which the money invested at simple interest has doubled itself the sum put out at *true* compound interest (*i.e.* interest collected and added to the capital every instant) will become **e** times its original value.

If the money had been invested at 5 per cent. compound interest per annum, it would become £100 **e** in 20 years (*i.e.* in $\frac{100}{5}$). If invested at 20 per cent. it would become £100 **e** in five years (*i.e.* in $\frac{100}{20}$), and in general, if invested at r per cent., it would become £100 **e** in $\dfrac{100}{r}$ years.

$$\therefore\quad £100 \text{ becomes } £100\,\mathbf{e} \text{ in } \frac{100}{r} \text{ years.}$$

$$\therefore\quad £100 \text{ becomes } £100\,\mathbf{e}^{\frac{r}{100}} \text{ in one year.}$$

$$\therefore\quad £100 \text{ becomes } £100\,\mathbf{e}^{\frac{rt}{100}} \text{ in } t \text{ years.}$$

The most general way of expressing this is as follows:—

If Q_0 = original quantity (*e.g.* the principal),
r = rate of growth per cent. **per unit of time** (*e.g.* rate of interest per cent. per annum),
t = number of such units of time during which the quantity is so allowed to grow **continuously** (*e.g.* number of years during which the capital is invested),

and Q_t = the amount to which Q_0 has grown in the time t,

then
$$Q_t = Q_0\mathbf{e}^{\frac{rt}{100}}.$$

This is one of the most fundamental formulæ with which we shall have to deal throughout this book, and the reader is most earnestly recommended to get a clear grasp of its meaning and commit it to memory.

Since we assume r to be constant during the period of growth,

$$\therefore \quad \frac{r}{100} \text{ is constant } = k \text{ (say).}$$

$$\therefore \quad \text{Equation } Q_t = Q_0 e^{\frac{r}{100}t} \text{ becomes}$$

$$Q_t = Q_0 e^{kt} \quad . \quad . \quad . \quad \text{(i)}$$

This, then, is a somewhat modified form of the same equation for the compound interest law.

[*Note.*—k is called the constant of increment.]

Further, by taking logarithms on both sides, we get

$$\log_{10} Q_t = \log_{10} Q_0 + kt \log_{10} e.$$

$$\therefore \quad \log_{10} Q_t - \log_{10} Q_0 = kt \log_{10} e$$

or

$$kt \log_{10} e = \log_{10} \frac{Q_t}{Q_0}.$$

But since

$$e = 2{\cdot}718\ .\ .\ .$$

$$\therefore \quad \log_{10} e = \log_{10} 2{\cdot}718 = 0{\cdot}4343.$$

$$\therefore \quad 0{\cdot}4343kt = \log_{10} \frac{Q_t}{Q_0}.$$

$$\therefore \quad k = \frac{1}{0{\cdot}4343t} \log_{10} \frac{Q_t}{Q_0}$$

$$= \frac{2{\cdot}303}{t} \log_{10} \frac{Q_t}{Q_0} \quad . \quad . \quad . \quad \text{(ii)}$$

Now, as Q_t represents the amount to which Q_0 has grown in time t, $\therefore$ $Q_t = Q_0 + x_t$ where x_t represents the increment during the time t.

$\therefore$ Finally, the most general way of expressing the compound interest law is $k = \dfrac{2{\cdot}303}{t} \log_{10} \dfrac{Q_0 + x_t}{Q_0}$.

(See pp. 82 and 85, and Chapter XV., p. 258 *et seq.*)

The equation for the compound interest law may therefore be expressed in either of two forms, viz.:

either (1) exponential form $Q_t = Q_0 e^{kt}$,

or (2) logarithmic form $k = \dfrac{2{\cdot}303}{t} \log_{10} \dfrac{Q_0 + x_t}{Q_0}$.

The second form is the one most commonly used in the literature, but sometimes it is more convenient to use the first form. Whichever formula the student uses he must never

forget what k stands for: it is the rate at which the magnitude increases or decreases in a unit of time.

The Distinguishing Peculiarity of the Series " e ".—The series $1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots$ has one primary remarkable property which renders it peculiarly adaptable to logarithmic computation. This peculiarity is that

$$e^x = 1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\ldots$$

where x is any number, whether positive or negative, and whether an integer or a fraction.

That this is so can be seen at once from the following considerations:—

$$\begin{aligned} e^x = \left[\left(1+\frac{1}{n}\right)^n\right]^x &= 1+\frac{nx}{1}\cdot\frac{1}{n}+\frac{nx(nx-1)}{1.2}\cdot\frac{1}{n^2} \\ &\quad+\frac{nx(nx-1)(nx-2)}{1.2.3}\cdot\frac{1}{n^3} \\ &\quad+\ldots \\ &= 1+\frac{nx}{1}\cdot\frac{1}{n}+\frac{n^2x^2}{1.2}\left(1-\frac{1}{nx}\right)\cdot\frac{1}{n^2} \\ &\quad+\frac{n^3x^3\left(1-\frac{1}{nx}\right)\left(1-\frac{2}{nx}\right)}{1.2.3}\cdot\frac{1}{n^3} \\ &\quad+\ldots \\ &= 1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\ldots \end{aligned}$$

when n is infinitely large. There is no other algebraical series which has the same property.

The series $$1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots$$

is called the **exponential series,** and the identity

$$e^x = 1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots$$

is called the *exponential theorem*.

By making $x = -1$, we get

$$e^{-1} = 1-\frac{1}{1}+\frac{1}{1.2}-\frac{1}{1.2.3}+\ldots$$

The exponential theorem, then, gives an expansion of e^x in terms of the index x. For another peculiarity, see p. 184.

Calculation of Logarithms by means of the Exponential Theorem.—As an illustration we will compute the logarithm of, say, 1·2. This is done by calculating it first to the base e, and then multiplying the result by the modulus 0·4343 to convert it into the logarithm of the number to the base 10.

Let
$$\log_e 1{\cdot}2 = m.$$
$$\therefore \quad 1{\cdot}2 = e^m.$$
$$\therefore \quad (1{\cdot}2)^x = e^{mx}.$$

Expanding $(1{\cdot}2)^x$, *i.e.* $(1+0{\cdot}2)^x$, by the binomial theorem (p. 66) and e^{mx} by the exponential theorem we get two series which are identically equal, viz.:

$$1 + \frac{x(0{\cdot}2)}{1} + \frac{x(x-1)(0{\cdot}2)^2}{1\,.\,2} + \frac{x(x-1)(x-2)(0{\cdot}2)^3}{1\,.\,2\,.\,3} + \ldots, \text{ which is } (1+0{\cdot}2)^x$$

$$= 1 + \frac{mx}{1} + \frac{m^2x^2}{1\,.\,2} + \frac{m^3x^3}{1\,.\,2\,.\,3} + \ldots, \text{ which is } e^{mx}.$$

$$\therefore \quad \frac{0{\cdot}2}{1} + \frac{(x-1)(0{\cdot}2)^2}{1\,.\,2} + \frac{(x-1)(x-2)(0{\cdot}2)^3}{1\,.\,2\,.\,3} + \ldots$$

$$= \frac{m}{1} + \frac{m^2x}{1\,.\,2} + \frac{m^3x^2}{1\,.\,2\,.\,3} + \ldots$$

As an identity this is true for all values of x, including $x = 0$. Putting $x = 0$, we get, for the first series,

$$\frac{0{\cdot}2}{1} - \frac{(0{\cdot}2)^2}{2} + \frac{(0{\cdot}2)^3}{3} - \frac{(0{\cdot}2)^4}{4} + \frac{(0{\cdot}2)^5}{5} - \ldots,$$

and the second series becomes m, *i.e.* $\log_e 1{\cdot}2$ (all other terms vanishing).

$$\therefore \quad \log_e 1{\cdot}2 = 0{\cdot}2 - \frac{0{\cdot}2^2}{2} + \frac{0{\cdot}2^3}{3} - \frac{0{\cdot}2^4}{4} + \ldots = 0{\cdot}1823.$$

$$\therefore \quad \log_{10} 1{\cdot}2 = 0{\cdot}4343\left(0{\cdot}2 - \frac{0{\cdot}2^2}{2} + \frac{0{\cdot}2^3}{3} - \frac{0{\cdot}2^4}{4} + \ldots\right)$$

$$= 0{\cdot}4343 \times 0{\cdot}1823 = 0{\cdot}0792.$$

Generally, $\log_e a = \dfrac{a-1}{1} - \dfrac{(a-1)^2}{2} + \dfrac{(a-1)^3}{3} - \dfrac{(a-1)^4}{4} + \ldots$

This is called the **logarithmic series.**
By putting $a = (1+x)$ the series becomes

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (Cf. \text{ Exercise (1), p. 314.})$$

The logarithmic series, as it stands, has a very limited scope, since for numbers higher than 2 (*i.e.* when x is greater than 1) the series is divergent and cannot therefore be summed at all, while for numbers between 0 and 2, unless $(a-1)$ is a small decimal (say, not higher than 0·5) the convergence is so small as to render the series of very little practical use. Thus:

$$\log_e 3 = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \ldots \text{ which is divergent}$$

and $\log_e 1{\cdot}8 = 0{\cdot}8 - \frac{0{\cdot}8^2}{2} + \frac{0{\cdot}8^3}{3} - \frac{0{\cdot}8^4}{4} + \ldots$ converges so slowly that more than 20 terms are needed to get a result correct to 4 decimal places. Nevertheless, the exercise of a little ingenuity enables one to use the series for computing any number. Let us take, for instance, the computation of $\log_{10} 2$.

Since $\quad 2 = 8^{\frac{1}{3}}, \quad \therefore \quad \log_{10} 2 = \frac{1}{3}\log_{10} 8.$

$$\text{But } \log_{10} 8 = 1 + \log_{10} 0{\cdot}8 = 1 + 0{\cdot}4343 \log_e 0{\cdot}8 = 1 + 0{\cdot}4343 \log_e (1 - 0{\cdot}2)$$

$$= 1 - 0{\cdot}4343\left(0{\cdot}2 + \frac{0{\cdot}2^2}{2} + \frac{0{\cdot}2^3}{3} + \frac{0{\cdot}2^4}{4} + \frac{0{\cdot}2^5}{5} + \ldots\right).$$

As $0{\cdot}4343 \times \frac{0{\cdot}2^5}{5} = 0{\cdot}00003$, the result will be correct to four decimal places if we stop at that term. We then have

$$\log_{10} 8 = 1 - 0{\cdot}4343 \times 0{\cdot}2231 = 0{\cdot}9031.$$

$$\therefore \quad \log_{10} 2 = \frac{0{\cdot}9031}{3} = 0{\cdot}3010.$$

$$\text{Similarly } \log_{10} 3 = \tfrac{1}{2}\log_{10} 9 = \tfrac{1}{2}(1 + \log_{10} 0{\cdot}9) = 0{\cdot}5 + \frac{0{\cdot}4343}{2}\log_{10}(1 - 0{\cdot}1)$$

$$= 0{\cdot}5 - \frac{0{\cdot}4343}{2}\left(0{\cdot}1 + \frac{0{\cdot}1^2}{2} + \frac{0{\cdot}1^3}{3} + \frac{0{\cdot}1^4}{4} + \ldots\right),$$

which to four terms of the series

$$= 0{\cdot}5 - \frac{0{\cdot}4343}{2} \times 0{\cdot}1054 = 0{\cdot}4771.$$

For the reason that Napierian logarithms, or logarithms to the base e, can be at once calculated by using the logarithmic series, they are also called *natural logarithms. The conversion figure 0·4343 is called the modulus of the common system of logarithms* and is frequently denoted by the letter μ.

Conversely, if we are given any formula in which the logarithms are expressed to the base e and we want to convert it to one with logarithms to the base 10, it is necessary to multiply by $\frac{1}{0{\cdot}4343}$, *i.e.* by 2·303 (more exactly 2·302585).

Thus, if $\quad k = \frac{1}{t}\log_e \frac{Q_0 + x_t}{Q_0}$

then $\quad k = \frac{2{\cdot}3}{t}\log_{10} \frac{Q_0 + x_t}{Q_0}$ (see p. 79).

Example.—The influence of temperature upon the velocity of protein digestion is given by the equation

$$\frac{k_2}{k_1} = e^{5285(T_2 - T_1)/T_1T_2} \quad (\textit{cf. } \text{p. 280 } \textit{et seq.})$$

where k_1 and k_2 are the velocities at the absolute temperatures T_1, T_2. (Absolute temperature T is $273+t$, where t is the ordinary temperature on the centigrade scale.) At what temperature will the velocity be double that at 22·6°?

$$\log\frac{k_2}{k_1} = 5285\frac{(T_2 - T_1)}{T_1T_2}\log e$$

$$= 0·4343 \times 5285\frac{(T_2 - T_1)}{T_1T_2}.$$

In our case $\qquad k_2 = 2k_1$ and $T_1 = 273 + 22·6$

$$= 295·6.$$

$$\therefore \quad \log 2 = 0·4343 \times 5285\frac{(T_2 - 295·6)}{295·6T_2},$$

or
$$0·30103 = \frac{0·4343 \times 5285(T_2 - 295·6)}{295·6T_2},$$

or
$$\frac{0·30103 \times 295·6}{0·4343 \times 5285}T_2 = T_2 - 295·6,$$

i.e.
$$0·039T_2 = T_2 - 295·6,$$

or
$$0·961T_2 = 295·6.$$

$$\therefore \quad T_2 = \frac{295·6}{0·961} = 307·6° \text{ absolute}$$

$$\text{and } t = 307·6 - 273 \text{ °C.}$$
$$= 34·6° \text{ C.}$$

From all that has been said in this chapter the reader will appreciate the importance of the series e, or of the peculiar number 2·71828. . . . The series not only affords a ready means of calculating logarithms, but it also occurs in the consideration of all natural phenomena which take place in accordance with the compound interest law (see Chapter VII.).

EXERCISES.

(1) Prove that $\dfrac{e+e^{-1}}{2} = 1 + \dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \ . \ . \ .$

(2) Find the value of $\left(1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ . \ . \ .\right)\left(1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \ . \ . \ .\right)$

[*Answer*, $e \times e^{-1} = 1$.]

(3) Prove that $(1 + x + x^2 + x^3 + \ . \ . \ .$ to infinity$)(1 - x + x^2 - x^3 + \ . \ . \ .$ to infinity), when $x < 1$, is $1 + x^2 + x^4 + x^6 + \ . \ . \ .$ to infinity.

$\left[\textit{Answer}, \left(\dfrac{1}{1-x}\right)\left(\dfrac{1}{1+x}\right) = \dfrac{1}{1-x^2}.\right]$

(4) Compute $\log_{10} 7$.

$\left[\textit{Answer}, 1 - 0·4343\left(0·3 + \dfrac{0·3^2}{2} + \dfrac{0·3^3}{3} + \ . \ . \ . \text{ to 6 terms}\right) = 0·8451.\right]$

The Meaning of e^{-1}.—Consider now a ease where a man who has a eapital Q_0 spends r per eent. (*i.e.* $r/100$) of it every year. Then, in $100/r$ years the eapital would eompletely vanish (*e.g.* if $r = 10$ per eent. the eapital would be gone in $10 = 100/10$ years; if $r = 5$ per eent. the money will disappear in $20 = 100/5$ years). The time taken for the eapital to vanish would be the same whether the man spent $r/100$ of it every year for $100/r$ years, or $r/1200$ of it every month for the $1200/r$ months of the $100/r$ years, or $r/36{,}500$ of it every day during the $36{,}500/r$ days of the $100/r$ years. But supposing the man were to spend $r/100$ of the eapital Q_0 at the end of the first year, leaving $Q_0\left(1 - \frac{r}{100}\right)$. Then at the end of the seeond year he were to spend $r/100$ *not of the original capital* Q_0 *but of the* $Q_0\left(1 - \frac{r}{100}\right)$ *left, i.e.* $Q_0\left(1 - \frac{r}{100}\right) \cdot \frac{r}{100}$; the amount left after that would be $Q_0\left(1 - \frac{r}{100}\right) - Q_0\left(1 - \frac{r}{100}\right) \cdot \frac{r}{100} = Q_0\left(1 - \frac{r}{100}\right)\left(1 - \frac{r}{100}\right) = Q_0\left(1 - \frac{r}{100}\right)^2$. If then at the end of the third year he were to spend $r/100$ *of the amount then available, i.e.* $r/100$ of the $Q_0\left(1 - \frac{r}{100}\right)^2$, the amount left would be $Q_0\left(1 - \frac{r}{100}\right)^3$, and so on; so that after $100/r$ years the amount left would be $Q_0\left(1 - \frac{r}{100}\right)^{100/r}$. If, however, instead of spending $r/100$ of the *available* eapital every year he spent $r/1200$ of the amount available every month, then after $100/r$ years the amount left would be $Q_0\left(1 - \frac{r}{1200}\right)^{1200/r}$. Similarly, if $r/36{,}500$ of the available amount were spent every day, the amount left after $100/r$ years would be $Q_0\left(1 - \frac{r}{36{,}500}\right)^{36500/r}$. Finally, if it were possible to remove $1/n$ of the eapital eontinuously every instant (n being infinitely large), then the amount left after $100/r$ years would be $Q_0\left(1 - \frac{1}{n}\right)^n$. When $\left(1 - \frac{1}{n}\right)^n$ is expanded by the binomial theorem, we obtain

$$\left(1 - \frac{1}{n}\right)^n = 1 - \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{n^2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{n^3} + \cdots$$

(see p. 66) which, in the manner shown in the case of $\left(1+\frac{1}{n}\right)^n$ (p. 77), becomes when $n = \infty$,

$$1 - \frac{1}{1} + \frac{1}{1\,.\,2} - \frac{1}{1\,.\,2\,.\,3} + \frac{1}{1\,.\,2\,.\,3\,.\,4} - \ . \ . \ .$$

But we have seen (p. 80) that this series $= \mathrm{e}^{-1}$. Hence we see that e^{-1} $\left(\text{or } \frac{1}{\mathrm{e}}\right)$ is the limit of $\left(1-\frac{1}{n}\right)^n$ when n becomes infinitely large $\left[\textit{symbolically}, \mathrm{e}^{-1} = \mathrm{Lt}_{n\to\infty}\left(1-\frac{1}{n}\right)^n\right]$, and represents the amount which a unit quantity *diminishing* continuously at a rate which is a constant proportion of its magnitude at every instant will reach after that period at which the same quantity would totally vanish if it diminished at the same rate but proportionally to its original value.

In general, if r is the percentage rate of decrease of Q_0 per unit of time, t is the number of such units of time during which Q_0 is so allowed to shrink continuously and Q_t is the amount to which Q_0 has shrunk in the time t, then

$$Q_t = Q_0\mathrm{e}^{\frac{-rt}{100}} = Q_0\mathrm{e}^{-kt},$$

where $k = r/100$ = the decrease per unit time, k being called the constant of diminution or of negative increment.

By taking logarithms of both sides (as on p. 79), we obtain

$$k = \frac{2{\cdot}303}{t}\log_{10}\frac{Q_0}{Q_t} = \frac{2{\cdot}303}{t}\log_{10}\frac{Q_0}{Q_0 - x_t},$$

where x_t is the amount of diminution during the time t.

Hence, whenever the reader comes across either of the following equations, viz.:

$$Q_t = Q_0 e^{kt}$$
$$Q_t = Q_0 e^{-kt}$$
$$k = \frac{1}{t}\log_e\frac{Q_0 + x_t}{Q_0}$$
$$k = \frac{1}{t}\log_e\frac{Q_0}{Q_0 - x_t}$$

he knows at once that he is dealing with an example of growth (positive or negative) in accordance with the compound interest law.

CHAPTER VII.

THE SIMPLE AND COMPOUND INTEREST LAWS IN NATURE.

(A) Examples of the Simple Interest Law.

Although, as we shall see later (p. 87 *et seq.*), many of the phenomena occurring in Nature take place in accordance with the law of compound interest, there are a good few examples of natural phenomena which, at any rate, as a first approximation obey the simple interest law.

(1) **The Coefficient of Expansion.**—Supposing a body such as a thin, long metal rod to be heated; then we know from elementary physics that if l_0 = length of rod at 0° C. and a = coefficient of expansion, then l_t, the length of the rod at $t°$ C. $= l_0(1 + at)$.

Thus, putting $t = 0, 1, 2, 3, 4$, etc., we get the length of the rod at these different temperatures as

$$l_0, l_0(1 + a), l_0(1 + 2a), l_0(1 + 3a), l_0(1 + 4a) \ldots$$

i.e. the terms of an arithmetical progression. In other words, the law of thermal increase in length is the same as the simple interest law. The same applies approximately to the laws of superficial and cubical expansion.

(2) **Henry's Law of Solubility of Gases.**—This law states that at constant temperature the weight of gas dissolved by a unit volume of liquid is proportional to the pressure.

Thus

$$W = mp,$$

where m = amount of gas dissolved at one atmosphere and W = amount of gas dissolved at p atmospheres.

(3) **Distance Covered by a Body moving with a Uniform Velocity.**—We know from elementary mechanics that

$$S = Vt,$$

where V is the velocity per unit of time and S is the distance from some fixed point covered in t units of time, measured from some fixed instant.

(B) Examples of the Compound Interest Law in Nature.

The form of growth in which the amount of increment or of decrement is at every instant proportional to the magnitude at that instant of that which is increasing or decreasing is of particularly frequent occurrence in Nature, and hence Lord Kelvin classified all these forms of natural growth as examples of "the compound interest law in Nature." The following are a few examples:—

(1) **The Growth of a Population.**—We have already seen on p. 13, Example (8), that the increase of a population—assuming the rate of growth to be constant, *i.e.* undisturbed by undue emigration, immigration, war, epidemics, etc., takes place in accordance with the law of compound interest. Now, suppose the population of a certain country to double itself in 100 years; what is the rate of growth, per annum, assuming it to be constant? If the population is a million at the beginning of the century, what will it be in 20 years from the beginning?

Here the most convenient equation to use is

$$Q_t = Q_0 e^{\frac{rt}{100}}.$$

$$\therefore \quad Q_{100} = Q_0 e^{\frac{r}{100} \times 100} = Q_0 e^r = 2Q_0 \text{ (by hypothesis).}$$

$$\therefore \quad e^r = 2.$$

$$\therefore \quad r \log e = \log 2.$$

$$\therefore \quad r = \frac{\log 2}{0{\cdot}4343} = \frac{0{\cdot}3010}{0{\cdot}4343}, \text{ whence } r = 0{\cdot}6931,$$

i.e. the increase of the population is at the rate of 0·6931 per cent., or at the rate of 6·931 per thousand, per year.

For 20 years' time:

$$\begin{aligned} Q_{20} = Q_0 e^{\frac{r}{100} \times 20} &= Q_0 e^{\frac{0{\cdot}6931 \times 20}{100}} \\ &= Q_0 e^{0{\cdot}1386} \\ &= 1{,}000{,}000\, e^{0{\cdot}1386}. \end{aligned}$$

$$\begin{aligned} \therefore \quad \log_{10} Q_{20} &= \log_{10} 1{,}000{,}000 + 0{\cdot}1386 \log_{10} e \\ &= 6 + 0{\cdot}1386 \times 0{\cdot}4343 \\ &= 6{\cdot}0602. \end{aligned}$$

$$\therefore \quad Q_{20} = 1{,}148{,}700.$$

The commonest way in which the compound interest law is met with in natural phenomena is when a magnitude **decreases** continually in proportion to its size at any moment.

The following are a few examples:—

(2) Arseniuretted hydrogen (AsH_3), when heated, splits up into arsenic and hydrogen in accordance with the equation

$$2AsH_3 = 2As + 3H_2.$$

Now, by Guldberg and Waage's law, the rate of decomposition at any moment is proportional to the active mass of the substance undergoing decomposition, *i.e.* to the amount of the substance present, at that moment.

Therefore: If 1 gramme of AsH_3 becomes, say, 0·9 gramme after one minute,

then the 0·9 gramme of AsH_3 becomes $(0{\cdot}9)(0{\cdot}9)$ gramme $= (0{\cdot}9)^2$ gramme after the second minute,

and the $0{\cdot}9^2$ gramme of AsH_3 becomes $(0{\cdot}9)^2(0{\cdot}9) = (0{\cdot}9)^3$ gramme after the third minute,

and the $0{\cdot}9^3$ gramme of AsH_3 becomes $(0{\cdot}9)^3(0{\cdot}9) = (0{\cdot}9)^4$ gramme after the fourth minute,

and so on.

Here, then, since the AsH_3 keeps on decomposing at every moment, the ultimate quantity left after ten minutes—when, diminishing in accordance with the simple interest law (*i.e.* if the amount of AsH_3 decomposed every instant would be replaced so as to make the active mass always equal to 1 gramme) the total amount of **original** AsH_3 would have entirely disappeared—is $\left(1 - \frac{1}{n}\right)^n$ where n is infinitely great. This, as we have seen on p. 85, is the same as e^{-1}, and hence we have that the amount Q_t of AsH_3 left at any moment (t) is Q_0e^{-kt}.

(3) A pane of glass obliterates 5 per cent. of the light falling upon it; how much light gets through twenty such panes, one behind the other, assuming that they all act in the same way?

$Q_t = Q_0e^{-kt}$. (Q_t = amount of light passing through t panes of glass.)

When $t = 1$ pane of glass, 5 per cent. of the light is obliterated, *i.e.* 95 per cent. of the light passes through.

$$\therefore \quad Q_1 = \tfrac{95}{100}Q_0.$$

But $\quad Q_1 = Q_0e^{-k.1} = Q_0e^{-k}.$

$$\therefore \quad e^{-k} = 0{\cdot}95,$$

i.e. $\quad \dfrac{1}{e^k} = 0{\cdot}95.$

$$\therefore \quad e^k = \frac{1}{0{\cdot}95} = 1{\cdot}053.$$

$$\therefore \quad k \log_{10} e = \log_{10} 1{\cdot}053 \text{ or } 0{\cdot}4343\,k = \log_{10} 1{\cdot}053 = 0{\cdot}0224.$$

$$\therefore \quad k = \frac{0{\cdot}0224}{0{\cdot}4343} = 0{\cdot}05.$$

$$Q_{20} = Q_0 e^{-20k}$$
$$= Q_0 e^{-1}.$$

$$\therefore \quad \frac{Q_0}{Q_{20}} = e^1 = 2{\cdot}718.$$

$$\therefore \quad \frac{Q_{20}}{Q_0} = \frac{1}{2{\cdot}718} = 0{\cdot}37.$$

$\therefore$ About 37 per cent. of the light passes through.

This is the principle of the spectrophotometer.

If the thickness of one layer of glass in the last example is 10 cm., what thickness of the same glass will reduce the intensity of the light to $\frac{1}{2}$?

Here $$Q_t = \tfrac{1}{2}Q_0.$$

$$\therefore \quad \tfrac{1}{2}Q_0 = Q_0 e^{-kt},$$

or $$\tfrac{1}{2} = e^{-kt},$$

or $$2 = e^{kt}.$$

But $$k = 0{\cdot}05.$$

$$\therefore \quad 2 = e^{0\cdot05t} = e^{\frac{t}{20}},$$

$$\therefore \quad \log_{10} 2 = \frac{t}{20} \log_{10} e = \frac{t}{20} \times 0{\cdot}4343.$$

$$\therefore \quad 2{\cdot}3 \log_{10} 2 = \tfrac{1}{20}t.$$

$$\therefore \quad t = 46 \log_{10} 2$$
$$= 46 \times 0{\cdot}3010$$
$$= 13{\cdot}85 \text{ layers}$$
$$= 138{\cdot}5 \text{ cm. thick.}$$

(4) Another most interesting example is **the rate of cicatrisation of a wound,** which has been shown by Carrel, Hartmann, Lecomte du Noüy and others, to follow the compound interest law (see *Journ. Exp. Med.*, vol. xxiv., 1916, and vol. xxvii., 1918, and p. 362 of this book).

Further Example.—The following results were found by Winkelman in the case of a cooling body :—

t (time).	θ (temperature).
0	18·9
7·40	16·9
15·85	14·9
25·35	12·9
36·65	10·9

Show that the rate of cooling agrees with the compound interest law.

If the law of cooling is the same as the compound interest law, then

$$\theta_t = \theta_0 e^{-kt}$$

$$\therefore \text{(i)} \qquad \theta_{7\cdot40} = \theta_0 e^{-kt}$$

i.e. $$16\cdot9 = 18\cdot9e^{-k\times7\cdot4}$$

i.e. $$\frac{18\cdot9}{16\cdot9} = e^{7\cdot4k}$$

i.e. $$1\cdot12 = e^{7\cdot4k}$$

$$\therefore \qquad 7\cdot4k = 2\cdot3 \log 1\cdot12 = 2\cdot3 \times 0\cdot0492 = 0\cdot1132$$

$$\therefore \qquad k = 1132/74000 = 0\cdot0153.$$

(ii) When $t = 15\cdot85$,

$$\theta_{15\cdot85} = 14\cdot9$$

$$\therefore \qquad 14\cdot9 = 18\cdot9e^{-15\cdot85k}$$

i.e. $$e^{15\cdot85k} = 1\cdot27$$

$$\therefore \qquad 15\cdot85k = 0\cdot2390, \text{ whence } k = 0\cdot0151.$$

(iii) When $t = 25\cdot35$,

$$\theta_{25\cdot35} = 12\cdot9$$

$$\therefore \qquad 12\cdot9 = 18\cdot9e^{-25\cdot35k}$$

$$\therefore \qquad 1\cdot47 = e^{25\cdot35k}$$

$$\therefore \qquad 25\cdot35k = 0\cdot3848$$

$$\therefore \qquad k = 0\cdot0152.$$

(iv) For $t = 36\cdot65$ we have

$$10\cdot9 = 18\cdot9e^{-36\cdot65k}$$

$$\therefore \qquad 1\cdot73 = e^{36\cdot65k}$$

$$\therefore \qquad 36\cdot65k = 0\cdot5474,$$

$$\text{whence} \quad k = \frac{0\cdot5474}{36\cdot65} = 0\cdot0149.$$

Hence we have the following results:—

Between the intervals	0	and	7·40,	$k = 0\cdot0153$.
,, ,,	0	,,	15·85,	$k = 0\cdot0151$.
,, ,,	0	,,	25·35,	$k = 0\cdot0152$.
,, ,,	0	,,	36·65,	$k = 0\cdot0149$.

In other words, within the limits of experimental error, on the assumption that the law of cooling follows the compound interest law, the value of k remains constant for the various temperature intervals.

Hence the assumption is most probably correct.

Note.—The student must not fall into the trap of believing that because the observed and calculated results agree, therefore the theory in question is necessarily true. For whilst it is the case that disagreement between the observed and calculated results is definite evidence against the theory in question, agreement between the two results is not absolute evidence in its favour. *Occasionally two or more different formulæ will give results all of which are in agreement with observation.*

The following example will illustrate this point. It is taken from Mellor's "*Higher Mathematics*" (Longmans, Green & Co.).

Dulong and Petit's formula for velocity of cooling is $V = b(c^\theta - 1)$, where $b = 2\cdot037$ and $c = 1\cdot0077$.

Stefan gives the formula:

$$V = a\{(273+\theta)^4 - (273)^4\}$$

where $a = 10^{-9} \times 16{\cdot}72$;

and yet the results as calculated by either of these totally different formulæ are practically identical and agree closely with the observed results.

Thus:

θ, Excess of Temp. of Body above that of Medium.	Velocity of Cooling.		
	Observed.	Calculated by the Formulæ of	
		Dulong and Petit.	Stefan.
220	8·81	8·89	8·92
200	7·40	7·34	7·42
180	6·10	6·03	6·09
160	4·89	4·87	4·93
140	3·88	3·89	3·92
120	3·02	3·05	3·05
100	2·30	2·33	2·30

The subject will be dealt with more fully in Chapters XX., p. 333, and XXII., p. 357.

EXAMPLES.

(1) A cup of tea the temperature of which five minutes ago was 100° above that of the surrounding objects is now 80° above them. What will be its temperature in another half-hour, assuming this to fall by the compound interest law?

In five minutes the difference of temperature was reduced from 100° to 80°.

$$\therefore \text{ From equation } Q_5 = Q_0 e^{-5k} \text{ we have}$$

$$80/100 = e^{-5k}$$

$$\therefore \quad 100/80 \text{ or } 1{\cdot}25 = e^{5k}$$

$$\therefore \quad \log 1{\cdot}25 = 5k \log e$$

$$\therefore \quad 2{\cdot}3 \log 1{\cdot}25 = 5k$$

$$i.e. \quad 5k = 2{\cdot}3 \times 0{\cdot}097 = 0{\cdot}2231.$$

In another half-hour, t will be 35 minutes from the beginning,

$$\therefore \quad Q_{35} = Q_0 e^{-35k} = Q_0 e^{-0{\cdot}2231 \times 7}$$

$$= Q_0 e^{-1{\cdot}5617}$$

$$\therefore \quad \frac{Q_0}{Q_{35}} = e^{1{\cdot}5617}$$

$$\therefore \quad \log \frac{Q_0}{Q_{35}} = 1{\cdot}5617 \times 0{\cdot}4343$$

$$= 0{\cdot}6782$$

$$\therefore \quad \frac{100}{Q_{35}} = 4{\cdot}767$$

$$\therefore \quad Q_{35} = \frac{100}{4{\cdot}77} = 21° \text{ nearly.}$$

(2) A most interesting instance of the compound interest law is the hydrolysis of cane sugar, and catalysis in general. The hydrolysis in the presence of acid, which is called inversion, may be represented as follows:—

$$\underbrace{C_{12}H_{22}O_{11}}_{\text{Cane sugar}} + H_2O = \underbrace{C_6H_{12}O_6}_{\text{Dextrose}} + \underbrace{C_6H_{12}O_6}_{\text{Lævulose}}.$$

The acid is found not to undergo any change of concentration during the reaction. Now the amount of water in which the cane sugar is dissolved being very great compared with the amount of the sugar, the change in the concentration of the water brought about by the hydrolysis is negligible, and hence the only change of concentration to be considered is that of the cane sugar. Hence we have

$$Q_t = Q_0 e^{-kt}.$$

(Q_0 stands for the concentration of cane sugar at the beginning and Q_t for that after the time t.)

The change of concentration can be followed by means of the polariscope, and in this way the following figures were obtained, at temperature 25° C. (Cane sugar is dextro-rotatory and the mixture of dextrose and lævulose is lævo-rotatory, so that the rotation steadily diminishes as the reaction progresses, and finally changes its sign):—

t (in mins.).	Angle of Rotation.
0	+25·16
56	+16·95
116	+10·38
176	+ 5·46
236	+ 1·85
371	− 3·28
∞	− 8·38

(+ stands for rotation to the right, and − for rotation to the left.)

The angle of rotation at time 0 is that obtained before the beginning of the hydrolysis; the angle at time ∞ is that obtained after complete inversion.

Now, since during the progress of the reaction the angle of rotation changed from 25·16 to the right at the beginning (when the concentration of cane sugar was Q_0) to 8·38 to the left (when the concentration of cane sugar was zero), *i.e.* a total change of 25·16 + 8·38 = 33·54; and since, also, the concentration of the cane sugar at any moment t is proportional to the angle of rotation,

∴ Q_0 is proportional to and may be represented by the angle

$$(25{\cdot}16 + 8{\cdot}38) = 33{\cdot}54,$$

and Q_{116} (say) is proportional to and may be represented by the angle

$$(10{\cdot}38 + 8{\cdot}38) = 18{\cdot}76.$$

Hence we can put $\quad Q_{116} = 18{\cdot}76$ and $Q_0 = 33{\cdot}54$

and equation $Q_t = Q_0 e^{-kt}$ becomes

$$18{\cdot}76 = 33{\cdot}54 e^{-116k}$$

$$\therefore \quad e^{-116k} = \frac{18{\cdot}76}{33{\cdot}54}$$

$$\therefore \quad e^{116k} = \frac{33{\cdot}54}{18{\cdot}76} = 1{\cdot}79.$$

$$\therefore \quad 116k = 2{\cdot}3 \log 1{\cdot}79 = 0{\cdot}5816.$$

$$\therefore \quad k = \frac{0{\cdot}5816}{116} = 0{\cdot}0050.$$

Similarly at 176 minutes the amount of "invert" sugar formed may be put $= 25{\cdot}16 - 5{\cdot}46 = 19{\cdot}70$.

$\therefore \quad Q_{176}$ may be put $= 33{\cdot}54 - 19{\cdot}70 = 13{\cdot}84$.

$$\therefore \quad 13{\cdot}84 = 33{\cdot}54 e^{-176k}.$$

$$\therefore \quad 176k = 2{\cdot}3 \log \frac{33{\cdot}54}{13{\cdot}84} = 2{\cdot}3 \log 2{\cdot}42 = 0{\cdot}8827.$$

$$\therefore \quad k = 0{\cdot}0050.$$

Similarly for the other values of t. The value of k will be found in all these cases to be in the neighbourhood of 0·005.

(3) Cohnheim studied the affinity of albumose for HCl in the following manner:—

Equal volumes (5 c.c.) of a 10 per cent. solution of cane sugar were respectively mixed with

(i) 5 c.c. of an HCl solution containing 0·05 gramme HCl,
(ii) 5 c.c. of a mixture of 0·025 gramme HCl and 0·25 gramme albumose.

The angle of rotation of each of the cane sugar solutions was found to be 4·422. After mixing them respectively as above they were kept for the same length of time (four hours) at the same temperature. At the end of that time the polariscope reading was 1·272 in case (i), and 3·072 in case (ii).

Find whether there has been any combination of albumose with HCl, and if so in what proportion.

The logarithmic form of the equation for the compound interest law gives

$$k = \frac{2{\cdot}3}{t} \log_{10} \frac{Q_0}{Q_t}$$ (where Q_0 and Q_t are the respective cane sugar concentrations at the beginning and at the end of the experiment).

Therefore (as in Example 2) we have:—

In case (i) $\quad Q_0 = 4{\cdot}422 \quad$ and $\quad Q_t = 1{\cdot}272$

$$\text{Whence} \quad k_1 = \frac{2{\cdot}3}{t} \log \frac{4{\cdot}422}{1{\cdot}272} = \frac{2{\cdot}3}{4} \times 0{\cdot}5411;$$

and in case (ii) $\quad Q_0 = 4{\cdot}422 \quad$ and $\quad Q_t = 3{\cdot}072$.

$$\text{Whence} \quad k_2 = \frac{2{\cdot}3}{t} \log \frac{4{\cdot}422}{3{\cdot}072} = \frac{2{\cdot}3}{4} \times 0{\cdot}1582$$

$$\therefore \quad \frac{k_1}{k_2} = \frac{0{\cdot}5411}{0{\cdot}1582} = 3{\cdot}42.$$

Assuming the inversion velocities, k_1 and k_2, to be proportional to the concentrations of the *reacting* HCl, it follows that as the reacting concentration was 0·05 gramme HCl per 5 c.c. in the first case, it must have been $\frac{0{\cdot}05}{3{\cdot}42} = 0{\cdot}0146$ gramme per 5 c.c. in the second case. As, however, the total amount of HCl in the 5 c.c. of the mixture of acid and albumose was 0·025 gramme, it is to be inferred that $0{\cdot}025 - 0{\cdot}0146 = 0{\cdot}0104$ gramme HCl must have combined with the 0·25 gramme albumose. Hence, the albumose combined with 4·16 per cent. of its weight of HCl ($0{\cdot}25 : 0{\cdot}0104 = 100 : 4{\cdot}16$).

Note.—It is of course conceivable that the retardation of the inversion in the second case was not due so much to the chemical combination of the albumose with HCl as to the mechanical action of the colloidal albumose particles which may have hindered the progress of the inversion; but experiment shows that such mechanical retardation may be excluded.

EXERCISES.

(1) If a hot body cools so that in 24 minutes its excess of temperature has fallen to half the initial amount, how long will it take to cool down to 1 per cent. of the original excess?

[Since $\frac{1}{2} = e^{-24k}$, $\therefore$ $2 = e^{24k}$, whence $k = 0{\cdot}0288$.

If t = time during which temperature has fallen to 1/100 of original excess, we have

$$\tfrac{1}{100} = e^{-kt} = e^{0{\cdot}0288 - t},$$

whence $t = 160$ minutes.]

(2) The pressure P_h of the atmosphere at an altitude h kilometres is given by $P_h = P_0 e^{-kh}$, P_0 being the pressure at sea-level (760 mm.). The pressures at 10, 20 and 50 kilometres being 199·2, 42·2 and 0·32 mm. respectively, find the mean value of k.

$$\left[\textit{Answer, } k = \frac{0{\cdot}134 + 0{\cdot}145 + 0{\cdot}155}{3} = 0{\cdot}144.\right]$$

(3) The quantity Q of a radio-active substance which has not yet undergone transformation is known to be related to the initial quantity Q_0 of the substance by the relation $Q = Q_0 e^{-\lambda t}$, where λ is a constant and t is the time in seconds from beginning of transformation. Find the "mean life" of thorium, and of radium A, being given that for thorium $\lambda = 5$, and for radium A, $\lambda = 3{\cdot}85 \times 10^{-3}$. (By the "mean life" is meant the time required to transform half the substance.)

[*Answer*, For thorium, 0·14 second, and for radium A, 3 minutes.]

(4) An electric current left to die out in a certain circuit drops to $1/e$ of its value in $\frac{1}{10}$ of a second; how long will it take to drop to a millionth of its value, assuming that it decreases at a rate proportional to itself?

$$\left[\frac{1}{e} = e^{-\frac{k}{10}} \quad \therefore \quad k = 10. \quad \text{Hence } 10^{-6} = e^{-10t}\right.$$

whence $t = 1{\cdot}4$ seconds nearly.]

(5) Cholera bacilli double themselves in number in 30 minutes. Find the number that one bacillus would give rise to in 24 hours.

[$2 = e^{30k}$, $\therefore$ $k = 0{\cdot}0231$.

$\therefore$ in 24 hours, number $= e^{1440 \times 0{\cdot}0231}$

$= 28 \times 10^{13}$.]

(See Example (8), p. 188.)

(6) In a series of experiments with anthrax spores treated with a 5 per cent. solution of phenol at a temperature of 20·2° C., Miss Chick found the following numbers of surviving bacteria present at the stated times. Examine these numbers in the light of Miss Chick's theory that the number of organisms destroyed by the disinfectant at any moment is proportional to the number of living organisms present at that moment.

t (hours) . .	0	0·5	1·5	2·7	5·95	25·6
n (number of surviving bacteria) .	434	410	351	331	241	28

[The numbers of surviving bacteria agree with those found by means of the formula

$$n_t = n_0 e^{-kt} \text{ or } k = \frac{2{\cdot}3}{t} \log_{10} \frac{n_0}{n_t}.$$

The mean value of k will be found to be 0·108.]

For further examples and exercises on the Compound Interest Law, see Chapters XV. and XXII.

CHAPTER VIII.

FUNCTIONS AND THEIR GRAPHICAL REPRESENTATION.

THE term **function** has a different meaning in mathematics from what it has in physiology. In mathematics we speak of one quantity being a function of another when the value of the first quantity depends upon that of the second.

Thus in all the cases we have been considering in Chapter VII., the value of Q keeps on changing from Q_0 to Q_t as t changes—whatever t happens to represent—hence we say that Q is a function of t.

For example, the amount to which a capital grows at simple or compound interest—**when the rate of interest is fixed**—depends upon the time t, *i.e.* upon the number of years or the other units of time during which the capital is allowed to grow, and hence we say that the amount to which a capital grows at a fixed rate of interest is a **function** of the time.

Again, the temperature of a cooling body is a **function** of the time during which cooling occurs because, **so long as the temperature of the surroundings is fixed and unaltered,** its temperature gradually decreases as time increases.

The hydrolysis of sugar, or, indeed, the amount of chemical transformation occurring in any chemical reaction, is a function of time.

The amount of light passing through a given transparent substance is a function of the thickness of that substance.

Indeed, we can multiply the number of examples of functions almost indefinitely. Every problem that one investigates in the laboratory is an example of a function. When we investigate *Boyle's law* we are dealing with a function, viz. when the temperature is unaltered, then either the pressure of the gas is a function of the volume or the volume is a function of the pressure; *Charles' law*, which says that the volume of a gas at fixed pressure depends upon the temperature, is an example of volume being a function of temperature; and so on.

Variables and Constants.—In the examples we have been

considering, and, indeed, in the case of any mathematical function, we deal with at least two quantities which keep on changing, provided other conditions remain the same, throughout the period of the experiment. For example, with money growing at interest the only quantities that keep on changing are the amount to which the capital has grown and the time during which the capital has been allowed to grow, provided the rate of interest remains the same. In the case of chemical transformations, the only quantities that keep on changing are the amount of substance transformed and the time during which the reaction has been allowed to proceed, provided the temperature of the reacting substances is kept unaltered. Again, in our case of light passing through a transparent substance, the quantities which kept on changing were the amount of light passing through and the thickness of the substance through which the light was passing, the nature of the transparent substance remaining the same.

Now, the quantities which keep on changing their value in any function are called **variables,** and the quantity or quantities which remain fixed for the duration of the process are called **constants.** Hence we say that in the case of money growing at fixed interest the variables are the amount and the time, whilst the constant is the rate of interest. In the case of light passing through the transparent medium, the variables were the amount of light transmitted and the thickness of the medium, the constant being the nature of the medium (*i.e.* whether glass, water, oil, etc.). In the case of chemical transformation, the variables are the amount of substance transformed and the length of time of the transformation, the constant being the temperature; and so on.

Dependent and Independent Variables.—Now, of the two variables, we speak of one of the variables being dependent upon the other, *e.g.* the amount to which the capital has grown depends upon the time, and so on.

The variable whose value at any time depends upon another variable is called a **dependent** variable; whilst the other variable, the variation of which determines the value of the dependent variable, is called an **independent** variable.

Thus, the amount in the case of money growing at interest is called the dependent variable, the time during which the money grows is the independent variable, whilst the rate of interest is the constant. Similarly, in the case of chemical reactions, the amount of substance transformed is the dependent variable, the period of transformation is the

independent variable, whilst the temperature of the reaction is the constant. And so on, for any of the other cases.

Examples of functions occurring in biological inquiries are the relationships existing between the sitting height of a person and his vital capacity, or between the sitting height and the circumference of his chest, etc. Thus, Dreyer has shown that if λ = sitting height in centimetres, then

$$\text{Vital capacity in c.c.} = \frac{\lambda^{2\cdot257}}{6\cdot1172} \text{ (in the case of males),}$$

$$\text{Chest circumference in centimetres} = \frac{\lambda^{1\cdot1442}}{2\cdot00148} \text{ (in the case of males),}$$

$$\text{Weight of person in grammes} = (0\cdot38025\lambda)^{1/0\cdot319}.$$

(See p. 12, Example (5).)

The surface S (in square decimetres) of the body is a function of the weight W (in kilograms), and $S = K\sqrt[3]{W^2}$ (where K is a constant).

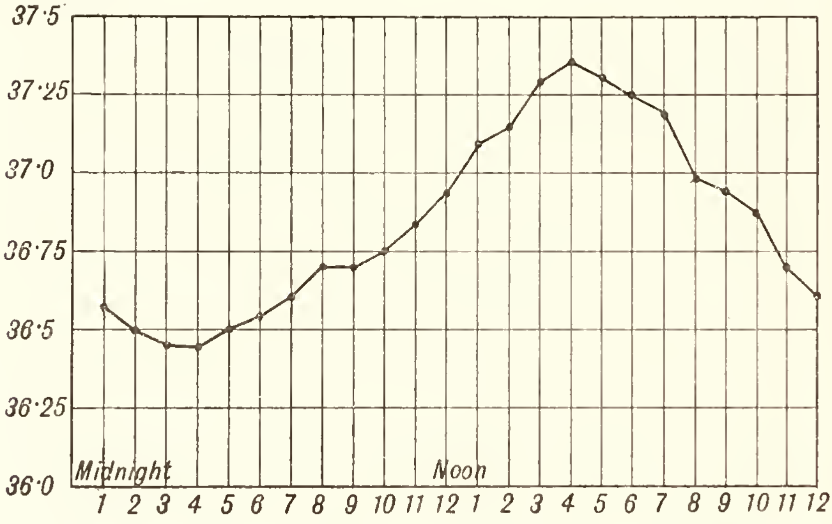

FIG. 25.—A One-Hourly Temperature Chart of a Normal Person.

Also the temperature of a person is a function of the time of day (fig. 25). The curve repeats itself every day, so that under normal conditions the temperatures of the same person are the same at corresponding hours of different days.

The pulse of a person is a function of the body temperature (other things remaining constant).

Definitions.—Hence we arrive at the following definitions:—

(*a*) *A constant* is a quantity which during any set of operations retains the same value.

(*b*) *A variable* is a quantity which during any set of operations keeps on changing in value.

(i) *A dependent variable* is one whose value at any moment depends upon the value of another variable at that moment.

(ii) *An independent variable* is one whose value at any moment determines the value of the dependent variable at that moment.

(*c*) *A function* is the numerical relation between one variable and another, *i.e.* the relation between the dependent and the independent variables, and is generally written, *e.g.*, as $y = f(x)$, which is read "y is a function of x."

One generally uses the last few letters of the alphabet to denote variables, *e.g.*, t, u, w, x, y, z, and the remaining letters, *e.g.*, a, b, c, or k, l, m, etc., to denote constants, so that in any function it is not necessary to point out in any other way which are the constants and which the variables.

Classes of Functions.—There are two main classes of functions, viz.:

(1) Algebraic. (2) Transcendental.

An **algebraic** function is one which contains the algebraic sum of some powers of x—whether those powers be positive or negative, integral or fractional; so that if $y = f(x)$, y can be expressed in terms of x by means of an equation consisting of a *finite* number of terms:

$$e.g., \quad y = x^5 + 7x^4 - x + 2.$$

A **transcendental** function is one which cannot be expressed in terms of x by an equation containing a limited or finite number of terms:

$$e.g., \quad y = \sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots \text{ to infinity}$$

(x being in radians).

Functions are also divided into

(1) Explicit. (2) Implicit.

An **explicit** function is one in which the dependent variable is expressed in terms of the independent variable:

e.g. $y = x^3 + 3x + 5$,
or $y = \log x$,
etc.

An **implicit** function is one in which the dependent and independent variables are involved together:

e.g. $y^2x - ayx + bx^2 + c = 0$.

Whilst an explicit function is written

$$y = f(x) \quad \text{or} \quad y = \phi(x), \text{ etc.},$$

an implicit function is written

$$f(y, x) = 0 \quad \text{or} \quad \phi(y, x) = 0, \text{ etc.}$$

The Graphical Representation of a Function.

The position of any point in a plane can be completely determined in one of the two following ways:—

(1) We can say that the point P is situated at certain distances NP, MP from two fixed straight lines Oy, Ox respectively in the same plane, crossing each other at right angles at the point O (fig. 26); or

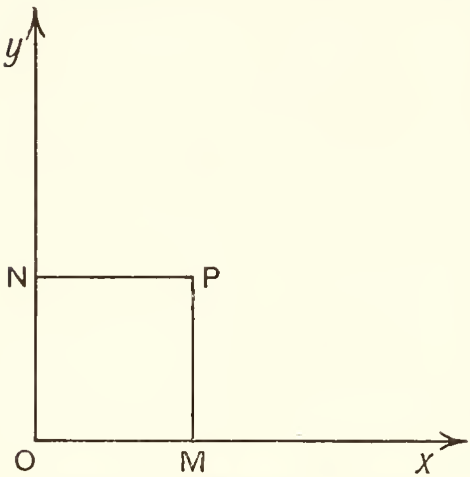

Fig. 26.—Cartesian Co-ordinates.

(2) We can say that the point P is situated at a certain distance OP from a fixed point O in the same plane, and that OP makes a certain angle θ with a fixed line Ox in the same plane (fig. 27).

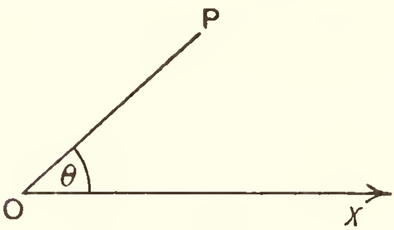

Fig. 27.—Polar Co-ordinates.

The first system of defining the position of a point is called the **Cartesian rectangular co-ordinate system** (after Réné Descartes, who invented it), whilst the second system is called the **polar co-ordinate system.**

The lines Ox, Oy in the rectangular co-ordinate system are called the **axes,** and O is called the **origin.** The horizontal line Ox is called the **abscissa axis.** The vertical line Oy is called the **ordinate axis.** The distances NP, MP of the point P from the two axes are called the *co-ordinates* of the point.

As NP = OM, which is a portion of the abscissa axis, and MP = ON, which is a portion of the ordinate axis, the co-ordinates NP and MP of the point P are called its abscissa and ordinate respectively.

The short way of indicating the position of the point P with reference to the axes is to call it the point (x, y), where x represents the length OM or NP (*i.e.* the abscissa), and y represents the length ON or MP (*i.e.* the ordinate).

Thus, if NP = 3 ft. and MP = 4 ft., P would be designated as the point (3, 4).

In the polar system, the point P would be referred to as the point (r, θ), where r = OP and $\theta = \angle POx$. Thus, if OP = 4 ft. and $\theta = 30°$, then the point P would be designated as the point (4, 30°).

In analytical geometry it is sometimes more convenient to use one system and sometimes the other.

By common convention it is agreed to call the direction along Ox, *i.e.* east of the origin, positive or +, and the direction along Ox' (fig. 28), *i.e.* west of the origin, negative or −.

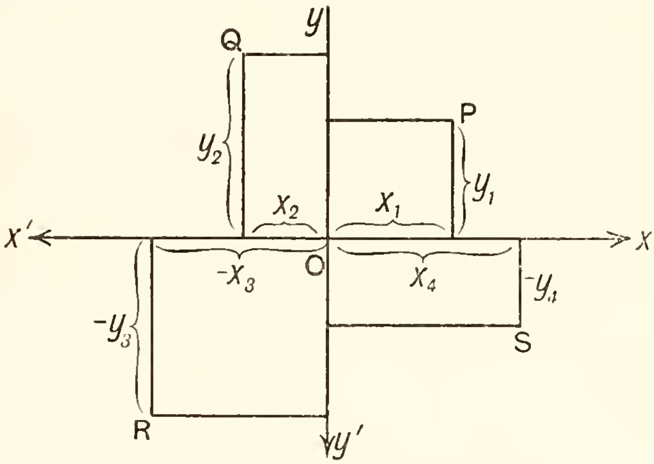

Fig. 28.—Convention with regard to Signs.

Also the direction along Oy, *i.e.* north of the origin, is called positive (+), and the direction along Oy', *i.e.* south of the origin, is called negative (−).

It must be remembered, however, that this form of designation is merely a convention, which may be departed from whenever convenience demands it. Thus, it may be found convenient to call the direction along Ox' or Oy' positive, and that along Ox or Oy negative. In such a case the sign would have to be specially indicated on the diagram. When there is no such indication it must always be taken for granted that the signs are in accordance with the convention.

Thus in fig. 28:—

The point P would be designated as the point (x_1, y_1).

,,	Q	,,	,,	,,	,,	$(-x_2, y_2)$.
,,	R	,,	,,	,,	,,	$(-x_3, -y_3)$.
,,	S	,,	,,	,,	,,	$(x_4, -y_4)$.

Hence, if we are given the co-ordinates of any point, we can easily locate its position. Thus, to find the position of the point (4, 2)—the units being in inches, for instance—we would measure off a distance OM (along Ox) = 4 in., and then measure off from M a distance of 2 in. along a line perpendicular to OM. The end of this line would give the required point P (see fig. 29).

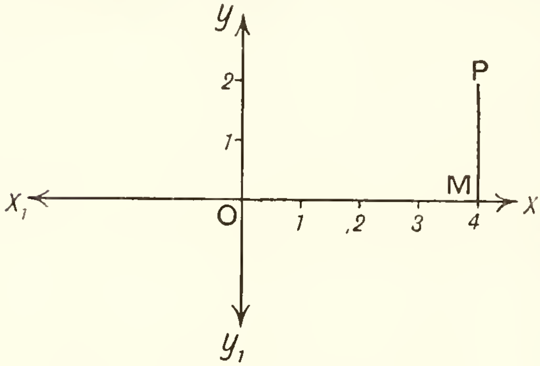

FIG. 29.—Position of a Point P in Space.

Anyone who has ever used a temperature or weight chart on which a patient's temperature or weight is recorded at fixed intervals on specially ruled paper is already acquainted with the use of co-ordinate geometry. On each of those charts successive equal portions of time (hours, or days, or other intervals of time, as the case may be) are represented by equal lengths measured, according to a certain scale, along a horizontal line (the x or abscissa axis), and degrees of temperature (above or below normal) or units of weight (above or below a certain selected weight) are represented by equal lengths measured either according to the same or a different scale, along a vertical line (the y or ordinate axis). The temperature or weight of the person at each fixed time is then recorded by placing a dot at the point which is vertically above or below the mark on the x axis corresponding to the particular time and at a height or depth equal to the length of the y axis corresponding to the particular temperature or weight. The line (irregular in these cases) joining the points obtained shows how the temperature or weight of the person varied during the period under consideration (see fig. 25 on p. 98).

In the same way one can record pictorially the relation between any two variables by representing successive equal values of the independent variable (x) by equal lengths measured along the x axis, and equal values of the dependent variable (y) by equal lengths measured along the y axis, **either on the same or a different scale.** By joining a number of points, so plotted, by means of a line (which may turn out to be straight or curved) one obtains a graphic representation of the particular

function, or the relationship between the dependent and independent variables.

Graph.—A straight or curved line (generically called a "curve"), the co-ordinates of every point on which satisfy the relation between two connected variables, is called a *graph*, and the process of drawing it is called *plotting* the graph.

Scales of Representation.—Whilst it is preferable and advisable to adopt, when possible, the *same* scale of representation in marking the units along the two axes (*e.g.* representing each unit along Ox and along Oy by $\frac{1}{10}$ in.), this is not always possible or convenient. In such cases *different* scales of representation must be adopted for the two axes. Thus a unit along the x axis may be represented by $\frac{1}{10}$ in., while that along the y axis may be represented on the same diagram by $\frac{2}{10}$, or $\frac{3}{10}$, or $\frac{5}{10}$, or $\frac{10}{10}$ in., and so on. The difference in scale must be allowed for in the course of any calculation in which the graph is used (figs. 30, 31, 33, 34, etc.).

EXAMPLE.

Plot a graph showing the relation between the weight and the height of women from the following plotting table:—

Weight in lbs. .	100	106	113	119	130	138	144
Height in inches .	60	61	62	63	64	65	66

The graph is shown in the diagram (fig. 30). Here the scale of representation adopted is such that each division along the x axis represents 5 lbs., and each division along the y axis represents 1 inch.

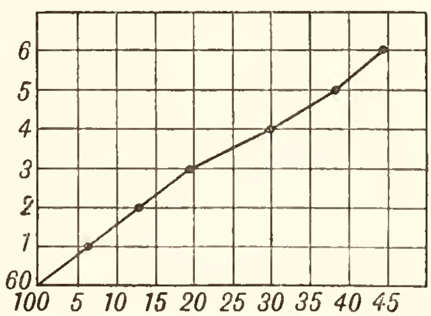

FIG. 30.—Graph showing Relation between Weight and Height of Women.

Different Kinds of Graphs.—(1) The points A, B, C, D, etc., in fig. 31 represent the weights in kilos of a certain infant at corresponding ages in weeks, and the thick line joining these

points is the graph showing the variation in weight (y) with age (x) between the ages of 28 and 44 weeks. Thus A represents

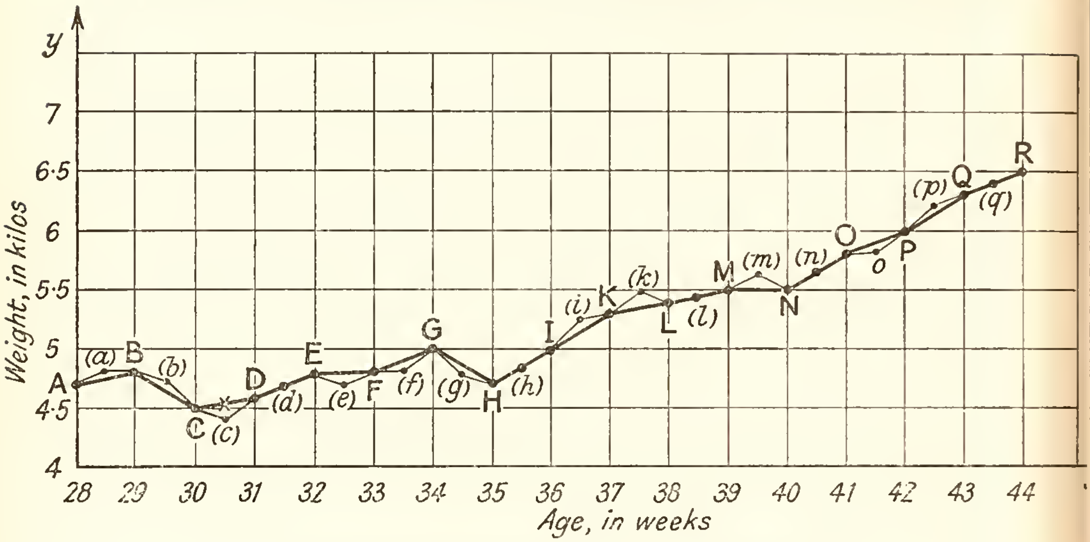

FIG. 31.—Weight-Age Graph of a Given Infant.

a weight of 4·7 kilos at the age of 28 weeks; B represents 4·8 kilos at 29 weeks; C represents 4·5 kilos at 30 weeks; and so on, up to R, which represents 6·5 kilos at 44 weeks.

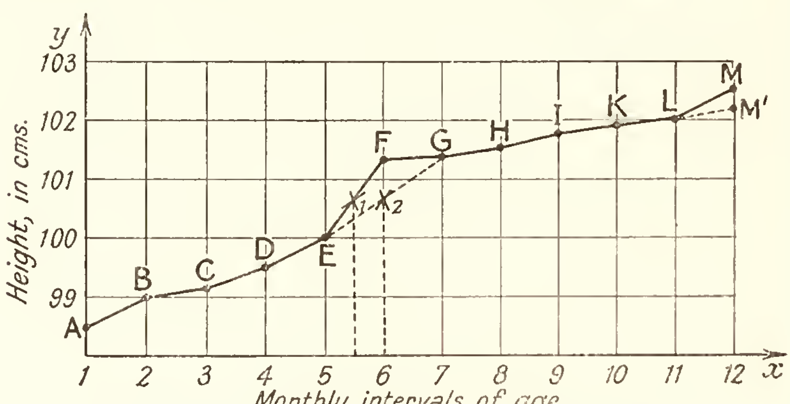

FIG. 32.—Height-Age Graph of a given Child between 4 and 5 Years Old.

(2) In fig. 32 the thick line is the graph drawn through the points A, B, C, D, etc., which represent the heights in centimetres of a certain child at monthly intervals between the ages of 4 and 5 years. Thus A represents 98·5 cm. at 4 years

1 month; B represents 99 cm. at 4 years 2 months; C represents 99·1 cm. at 4 years 3 months; and so on, up to M, which represents a height of 102·5 cm. at 5 years. The graph therefore represents the variation of the child's height (y) with age (x).

(3) In fig. 33 the thick line through A, B, C, D, etc. is the graph showing the variation in the area of a circle (y) with the length of its radius (x). Thus A is the area, 28·27 sq. cm., of a circle of radius 3 cm.; B is the area, 50·27 sq. cm., of a circle of radius 4 cm.; C is the area, 78·54 sq. cm., of a circle of radius 5 cm.; and so on, up to F, which is the area, 201 sq. cm., of a circle of radius 8 cm.

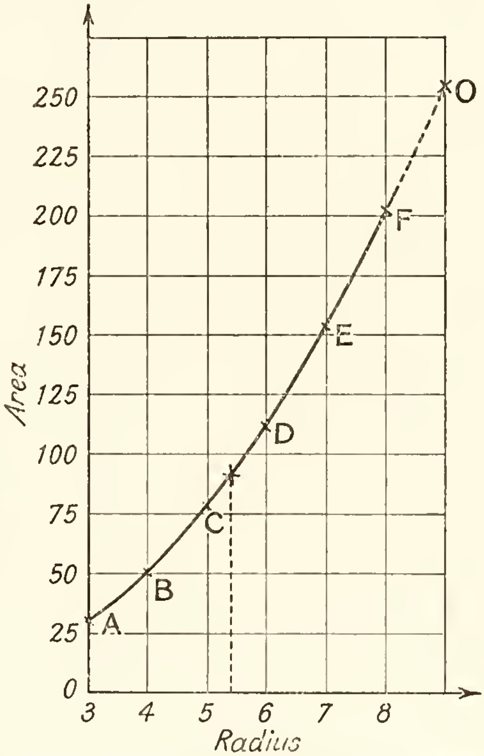

FIG. 33.—Graph showing Relation between Area of a Circle and its Radius.

A mere glance at these graphs at once reveals certain striking differences between them. Thus in fig. 33, *where the relation between y and x is a simple mathematical one,* viz. $y = \frac{22}{7}x^2$, *the graph is a perfectly continuous, smooth and regular curve.* Hence, if one were to be given a portion only of the graph, say that between B and E, it would be possible not only to *interpolate*, *i.e.* to infer the value of y for any value of x intermediate between any of the values represented by the points B and C, or C and D, etc., but it would be equally possible to continue the curve to any extent in either direction and thus to *extrapolate*, *i.e.* to estimate with absolute certainty the value—to any desired degree of accuracy—of y for any given value of x outside the limits represented by A and F. Thus, if we wish to find the area of a circle of radius 5·4 cm. we take a point X on the graph whose abscissa is 5·4 cm., and its ordinate—which is seen to be in the neighbourhood of 92 —represents the area, viz. about 92 sq. cm. [actual area $= \frac{22}{7}(5·4)^2 = 91·6$ sq. cm.] on the graph. Similarly the area

of a circle whose radius is 9 cm. is found by continuing the curve until it cuts the ordinate drawn from the abscissa $x = 9$; the height of that ordinate, which is seen to be about 255, represents the area of such a circle [actual area $= \frac{22}{7}(9)^2 = 254{\cdot}6$ sq. cm.]. The point is marked **O** on the dotted portion of the curve. The accuracy of the estimate varies with the scale on which the graph is drawn—the larger the scale the greater the accuracy.

Comparing fig. 32 with fig. 33 we see that the two graphs agree in so far that in both cases every increase in value of x (age in one case and radius in the other case) causes an increase in value of y (height and area respectively). There is, however, this profound difference between them, viz. that while in the case of the circle the *rate* of increase of y is perfectly regular, the same is not the case with age and height, and it is not possible to interpolate, with the same degree of accuracy, the height of the child for some age intermediate between two given ages as it is in the case of the area of a circle of radius intermediate between two given radii. It is, however, possible to make a fairly approximate estimate provided the two ages are not too far apart, because between the two ages we know that the height (y) must have risen—or possibly remained stationary—but could not have become less. Thus at the point $\mathbf{X}_1$ on the graph, corresponding to the middle of the sixth month, we may take it that the child's height was somewhere about midway between 100 cm. (height at end of fifth month) and 101·3 cm. (height at the end of the sixth month), *i.e.* approximately 100·7 cm. Owing, however, to the irregular rate of increase of height it would be impossible to estimate what would be the height of the child on either side of the limits of age between which the graph was drawn. Thus, if our data extended only up to the age of 4 years and 11 months we could not, from inspection of the graph, foretell by extrapolation that the child's height at the end of the 12th month would be represented by M, corresponding to 102·5 cm. (which actually was the height at that age), or by M′, corresponding to 102·2 cm., or any other level. Further, even with interpolation, if we did not know the child's height at the age of 4 years 6 months we could not estimate it even approximately from a knowledge of the heights at 4 years 5 months and 4 years 7 months—the age interval being too long. Thus, the interpolation value $\mathbf{X}_2$ found by taking the middle point of the line joining E and G represents a height of 100·7 cm., whereas the actual height at that age was 101·3 cm. (the point F).

Lastly, the graph of fig. 31 is completely irregular. Not only is there no regular rate of increase in the weight of any particular child with every unit increase in age, but there is not even an uninterrupted increase in weight, the y ordinate (weight) sometimes rising and sometimes falling. It is, therefore, impossible at any particular age to foretell what is likely to be the baby's approximate weight a week later, or even whether the weight will increase, remain stationary, or diminish. It is similarly impossible to infer with any degree of confidence what the baby's weight was at some fixed date intermediate between the dates of any two adjacent recorded dates. Hence, in a graph of this kind, it is not only impossible to extrapolate, but it is almost equally impossible to interpolate. Thus at 30·5 weeks, interpolation (X) gives the weight as 4·55 kilos, whereas actually the baby's weight at that time was 4·4 kilos (as shown at c). Similarly for other mid-intervals, as shown by the thin graph a, b, c, d, e, f . . .

Theorems.—What has been said about the regular graph of fig. 33 applies to all graphs plotted from a simple mathematical equation or function. Indeed, the following theorems are universally true:—

(1) *Every graph plotted from a given mathematical function gives a perfectly continuous, smooth and regular curve.*

(2) *Conversely, every plotted graph which is continuous, smooth and regular, can be represented by some mathematical function representing the relation between the two variables.*

(3) *If a plotted graph is more irregular than can be accounted for by errors of observation, it is quite certain that there is no simple mathematical function connecting the variables under investigation.*

In a future chapter we shall consider the method of ascertaining whether or not any irregularities in the shape of a graph are due to chance (*i.e.* fortuitous errors of observation), and how to assign a mathematical formula to any regular curve.

Once a simple mathematical relationship has been discovered and found to hold good for all observations repeated under the same conditions for wide ranges of values of the independent variable, then—and then only—is it justifiable to assume that the function so discovered is probably a correct representation of the relationship between the dependent and independent variables. If, in addition, the mathematical formula when used for purposes of extrapolation always gives correct results,

as subsequently confirmed by observation, so that it can, with confidence, be utilised for the purpose of predicting the results of any future observations for all values of the independent variable, then one is justified in claiming that the mathematical function so established constitutes a **Universal Law** for the relationship between the two variables. Thus Newton's formula $\frac{m_1 m_2}{r^2}$, which gives the gravitational attraction between two bodies of masses m_1 and m_2 separated by a distance r, having been found to answer all the foregoing tests, can with certainty be accepted as universally true—at any rate within the solar system—and is, therefore, truly said to express the *Law of Gravitation*. When such a law is established, it may be used to detect the nature of the disturbing factor in any instance where there is any divergence between observed and calculated values of the dependent variable. The discovery of Neptune, as the result of such discrepancies between the calculated and observed orbits of Uranus, constitutes the greatest glory of Newton's law.

The student will notice that the words "mathematical function" have been qualified by the adjective "simple." This is because, as will be seen later (p. 364), it is always possible to find some mathematical equation of the form $y = a + bx + cx^2 + dx^3 + \ldots + mx^n$ to fit any graph, provided the equation contains a sufficient number of terms, with as many constants as there are data in the plotting table (*i.e.* the table giving the values of y corresponding to different values of x). Such an equation, however, is mere pedantic camouflage, and inasmuch as it may deceive the untrained into believing that the phenomenon in question possesses the dignity of a mathematical law is to a certain extent dishonest. An equation of this type will, however, at once break down when subjected to the acid test of extrapolation. Nature's laws are always simple. The path of a projectile is a simple parabola; the orbit of a planet is an ellipse; and the behaviour of a gas under pressure conforms with the law of a hyperbola; in all cases these curves belong to the simple group of conic sections. The compound interest law, the law of inverse squares—of which the law of gravitation is an example—the law of harmonic motion, etc. are all expressible by simple mathematical formulæ which plot into simple curves. If one wishes to be facetious, one may say that "to be natural one must be simple!"

It is to be noted that two different formulæ may plot into the same curve within certain ranges of the variables (*e.g.* the formulæ for cooling bodies on pp. 90, 91 and those in the case of peptic digestion on pp. 333, 334. It is only by the *extrapolation test*, *i.e.* by the ability to predict correctly the numerical results of observations outside the given range, that one can decide between the relative merits of two rival formulæ).

Classification of Graphs.

What we have said enables us to divide graphs into:

(1) **Regular Graphs,** for which the relationship between the two variables can be expressed by some simple mathematical equation which can be used for both interpolation and extrapolation.

(2) **Irregular Graphs,** for which the relationship between the variables cannot be expressed by any simple mathematical equation. These may be subdivided into:

(a) *Completely Irregular Graphs*, for which y sometimes increases and sometimes decreases quite haphazardly with increase of x, and which it is impossible not only to extrapolate but even to interpolate.

(b) *Incompletely Irregular Graphs*, for which y keeps on increasing (or decreasing) steadily with increase of x, but the rate of that positive or negative growth is not regular. In these graphs it is possible to interpolate with some degree of accuracy between two values of one or the other variable not too far apart, but it is impossible to extrapolate.

The graph of fig. 31 belongs to type (a). For although *on the average* the weights of infants steadily increase with age and the rate of increase is such as to give rise to a fairly smooth and regular curve which can be represented by a simple mathematical equation (see p. 350), the same is not the case with any *particular* infant, whose weight may be affected in a positive or negative direction by a number of irregular environmental factors, such as the quality or quantity of its food, its clothing, the time of the year, its surroundings with respect to air, sunlight, etc. Similarly, a chart showing the monthly mortality rates of children from certain infectious diseases—say, diphtheria—at a given locality is a completely irregular graph, because the rates depend upon a number of uncertain factors, such as climatic conditions, the presence of an epidemic and its severity, the number of susceptible children, *i.e.* those who have not previously had the disease or been artificially immunised, etc.

The graph of fig. 32 is incompletely irregular, belonging to type (b), because while a child never decreases in height with increase of age, the amount of increase of height in any particular child during any interval depends upon a variety of factors similar to those which determine the change in the child's weight. It is probable, however, that *on the average* children grow in height in accordance with some mathematical

equation which, when plotted, will result in a *regular* graph. The increase in population during a period of years also gives rise to an incompletely irregular graph (fig. 34), for although in theory population naturally tends to grow in accordance with the compound interest law, in practice such growth is subject to numerous fluctuations due to such factors as immigration and emigration, epidemics, wars, birth control, etc.

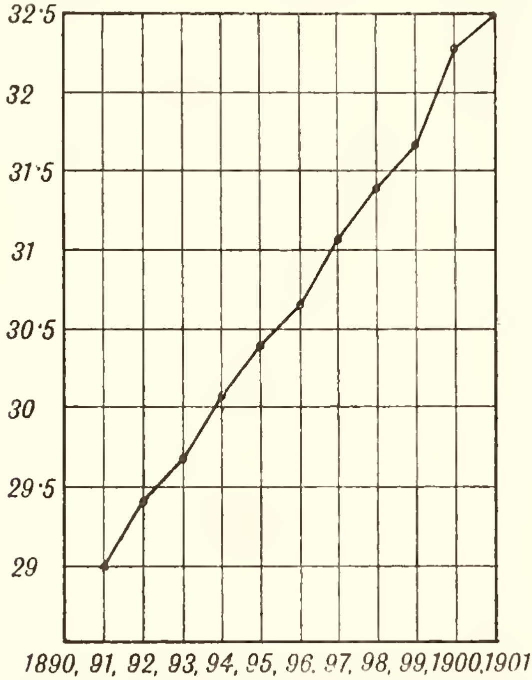

FIG. 34.—Graph of the Population in England and Wales between 1890 and 1901. Each yearly interval is represented by one division along the x-axis and each half-a-million population by one division on a different scale along the y-axis.

Types of Regular Graphs.—Whilst every particular function has its own particular graph, each **group** of functions of the same kind corresponds to one particular **type** of curve, which types we shall investigate in the course of the next few paragraphs. Thus, every function of the first degree in x and y is represented by a straight line, and is therefore called a **linear** function. A function of the type $y = x^2$ or $(x+a)^2$ is represented by a parabola; and so on.

Graph of an Equation of the First Degree in x and y.—Let us take a function like

$$y = 2x+3.$$

In order to draw its graph we proceed as follows (see fig. 35):—

(i) Give to x a series of values, say, 0, 1, 2, 3 . . .

(ii) Calculate the corresponding values of y, thus:

For $x = 0$,	$y = 2\times 0+3 = 3$,	$\therefore$ graph contains point	(0, 3)
,, $x = 1$,	$y = 2\times 1+3 = 5$,	$\therefore$,, ,,	(1, 5)
,, $x = 2$,	$y = 2\times 2+3 = 7$,	$\therefore$,, ,,	(2, 7)
,, $x = 3$,	$y = 2\times 3+3 = 9$,	$\therefore$,, ,,	(3, 9)
,, $x = 4$,	$y = 2\times 4+3 = 11$,	$\therefore$,, ,,	(4, 11)

etc., etc., etc.

Lastly, when $y = 0$, $2x+3 = 0$ and $\therefore$ $x = -\frac{3}{2}$.

(iii) Plot the various points (0, 3), (1, 5), (2, 7), etc., thus found.

(iv) Join these points by a "curve" passing through them.

It will be seen by inspection (and it is easily proved by the most rudimentary geometry) that the resulting graph is a straight line, possessing the following properties, viz.:

(1) It is inclined to the axis of x at an angle whose tangent is 2, *i.e.* the coefficient of x (thus, $AA'/O'A' = BB'/O'B'$, etc. $=2$).

(2) It cuts the y axis at a distance of 3 (*i.e.* the value of the term not containing a variable) from the origin.

(3) It cuts the x axis at a distance of $-\frac{3}{2}$ from the origin.

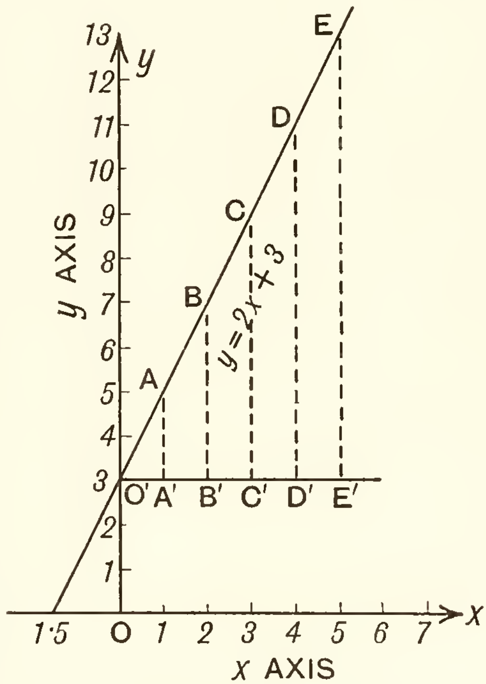

FIG. 35.—Graph of Function $y = 2x+3$.

Similarly, we can draw the graph of such a function as

$$2y = 3x-1 \quad \text{(see fig. 36).}$$

Thus, dividing by 2 we get $y = \frac{3}{2}x-\frac{1}{2}$.

For $x = 0$,	$y = -\frac{1}{2}$,	$\therefore$ graph contains the point	$(0, -\frac{1}{2})$.
,, $x = 1$,	$y = 1$,	$\therefore$,, ,, ,,	(1, 1).
,, $x = 2$,	$y = 2\frac{1}{2}$,	$\therefore$,, ,, ,,	$(2, 2\frac{1}{2})$.

etc., etc.

Lastly, when $y = 0$, $x = \frac{1}{3}$.

The "curve" passing through the various points thus found will again be found to be a straight line, whose properties are:

(1) It is inclined to the axis of x at angle whose tangent is $\frac{3}{2}$ (*i.e.* the coefficient of x).

(2) It cuts the axis of y at a distance equal to $-\frac{1}{2}$ (*i.e.* the value of the term not containing x) from the origin.

(3) It cuts the axis of x at a distance $\frac{1}{3}$ from the origin.

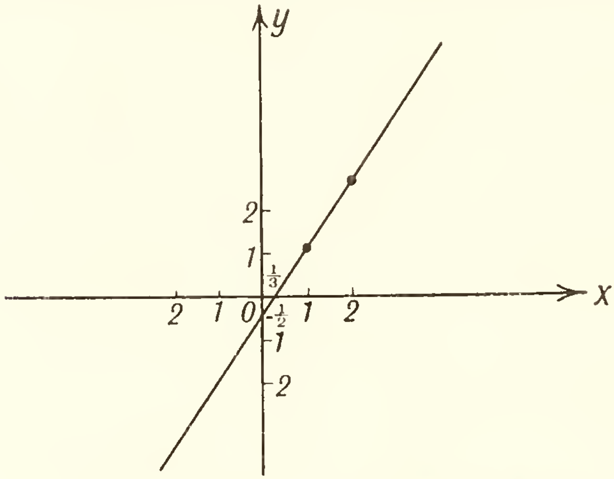

FIG. 36.—Graph of Function $2y = 3x - 1$, or $y = \frac{3}{2}x - \frac{1}{2}$.

Let us take one more example:

$$y = -\tfrac{2}{3}x + 1.$$

When	$x = 0,$	$y = +1.$
,,	$x = 1,$	$y = +\frac{1}{3}.$
,,	$x = 2,$	$y = -\frac{1}{3}.$
,,	$x = 3,$	$y = -1.$
Lastly, when	$y = 0,$	$x = +\frac{3}{2}.$

The graph (fig. 37) will again be found to be a straight line possessing the following properties:—

(1) Its inclination to the axis of x is $\tan^{-1}(-\frac{2}{3})$.
(2) Its y intercept $= 1$ unit.
(3) Its x intercept $= 1\frac{1}{2}$ units.

In general—

The equation $y = mx + b$ (i.e. any function $y = f(x)$ of the first degree in x and y) represents a straight line of unlimited length whose properties are that:

(1) **Its inclination to the x axis is $\tan^{-1} m$.**
(2) **Its y intercept is b units.**
(3) **Its x intercept is $-\frac{b}{m}$ units.**

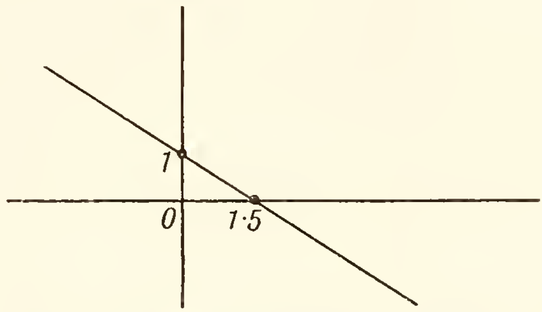

FIG. 37.—Graph of Function $y = -\frac{2}{3}x+1$, or $3y = -2x+3$.

From this there result the following five **corollaries**, viz.:—

(a) If $b = 0$, then, as there are no y and x intercepts, the line passes through the origin; *e.g.* the accompanying graphs (fig. 38) represent the lines

$$y = 2x; \quad y = x; \quad \text{and} \quad y = -\frac{x}{2}.$$

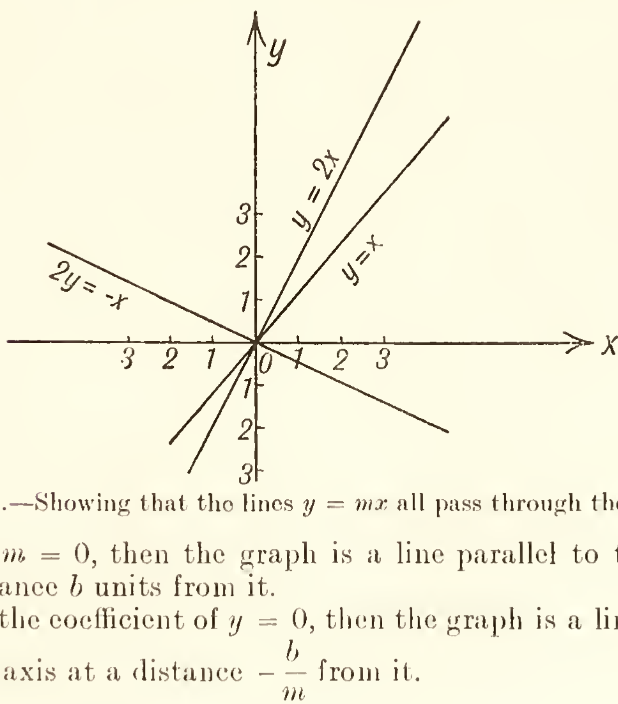

FIG. 38.—Showing that the lines $y = mx$ all pass through the Origin.

(b) If $m = 0$, then the graph is a line parallel to the x axis at a distance b units from it.

(c) If the coefficient of $y = 0$, then the graph is a line parallel to the y axis at a distance $-\frac{b}{m}$ from it.

(*d*) **All equations of the first degree in which the coefficients of y and the coefficients of x are respectively equal are represented by lines which are parallel to one another,** because, m being the same in all, their inclinations to the x axis are the same, viz. $\tan^{-1} m$. Similarly, **all lines in which the relationship between the coefficients of x and y is the same are parallel to one another.** Thus $y = 3x + 8$; $y = 3x + 5$; $y = 3x + \frac{3}{2}$ (or $2y = 6x + 3$); $y = 3x$; etc., are parallel (see fig. 39).

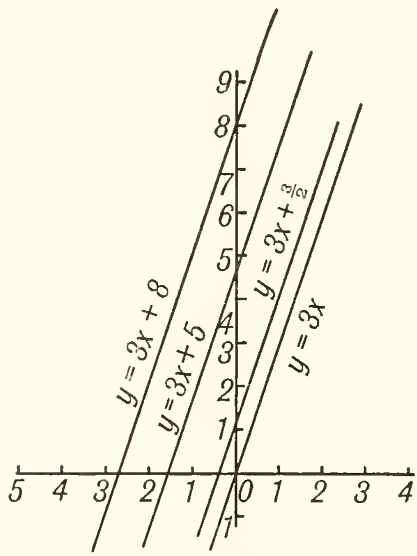

Fig. 39.—Showing that if the Value of m is constant, then the Lines are parallel.

(*e*) **To draw the graph of a function $y = mx + b$, it is sufficient to plot two points only,** since two points are sufficient to determine any straight line.

The most convenient points to plot are those found by making $x = 0$ and $y = 0$, viz. the points $(0, b)$ and $\left(-\frac{b}{m}, 0\right)$.

We see, therefore, that **when two variables are so related that one varies uniformly with the other, the graph of the function is a straight line.**

Hence all functions which are examples of the simple interest law may be represented as some straight line and are, therefore, as we have already said, called *linear functions*.

Examples of Linear Functions are (1) Henry's law of the solubility of gases, which states that the amount of gas which can be dissolved in a liquid at constant temperature is proportional to the partial pressure of the gas.

(2) The law of expansion of solids, liquids and gases, which states that the amount of expansion, whether linear, superficial, or cubical, is proportional to the temperature change.

(3) The law of motion at uniform velocity, which states that the distance covered is proportional to the time.

In many cases certain other functions may by logarithmic manipulation be converted into linear functions. As an example we may take Boyle's law of gases. This, although as we shall see later is a function whose curve constitutes a rectangular hyperbola, may yet be transformed in such a way as

to be capable of being represented by a straight line. Thus the law states that

$$PV = K$$

(P = pressure; V = volume; and K = constant).

By taking logarithms we have

$$\log P + \log V = \log K.$$

Hence, putting $\log P = y$

$\log V = x$

and $\log K = c$

we have $y = -x + c$,

which represents a straight line.

Similarly, the relation between H-ion concentration $[H^+]$ and OH-ion concentration $[OH^-]$ in water, represented by $[H^+][OH^-] = K$, can be transformed into a linear function by taking logarithms of each side:

$$\log [H^+] + \log [OH^-] = \log K$$

See Example (3), p. 358 *et seq.*

EXAMPLES.

(1) If we draw a straight line graph representing the equation $y = \frac{9}{5}x + 32$, we shall get a diagram representing the relation between the centigrade (x) and Fahrenheit (y) scales of temperature. To draw such a graph it is sufficient to plot two points only, *e.g.* the freezing-point (0, 32) and the boiling-point (100, 212), and join them by a straight line.

(2) Sörensen's method of expressing H-ion concentration is to give the minus logarithm of the H-ion concentration, *i.e.* $pH = -\log C_H$. Hence by plotting C_H or $[H^+]$ against pH or $\log [H^+]$, we get a graph for converting one into the other (H. E. Roaf).

(3) The temperature of a person was taken at two different times on Fahrenheit and Réaumur thermometers simultaneously, and found to be as follows:—

(1) Fahrenheit (F.°) 101·3, Réaumur (R.°) 30·8
(2) ,, ,, 102·2, ,, ,, 31·2.

(*a*) Assuming the relationship between the two scales to be linear, draw the graph and find its equation; (*b*) find by calculation and extrapolation the Fahrenheit temperature corresponding to 29·6° R.

(*a*) Let the R. scale be the x axis and the F. scale the y axis. The first pair of corresponding temperatures becomes the point (30·8, 101·3); the second pair of corresponding temperatures becomes the point (31·2, 102·2).

The graph must be a straight line passing through these points (the relationship between the variables being linear). The graph is shown in fig. 40.

Let its equation be $y = mx + b$.

Substituting each pair of values, we get

$$101{\cdot}3 = 30{\cdot}8m + b \quad . \quad . \quad . \quad . \quad (1)$$

$$102{\cdot}2 = 31{\cdot}2m + b \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2) we get

$$0{\cdot}9 = 0{\cdot}4m, \quad \therefore \quad m = \tfrac{9}{4}.$$

Put this value of m in equation (1), and we get

$$101{\cdot}3 = 30{\cdot}8 \times \tfrac{9}{4} + b = 69{\cdot}3 + b$$

$$\therefore \qquad b = 32.$$

$\therefore$ Equation of graph is $y = \frac{9}{4}x + 32$ (or F.° $= \frac{9}{4}$R.° $+ 32$).

(*b*) Putting x (or R.°) $= 29{\cdot}6$, we get y (or F.°) $= \frac{9}{4} \times 29{\cdot}6 + 32 = 98{\cdot}6$. Extrapolation gives the same value (point C on the graph).

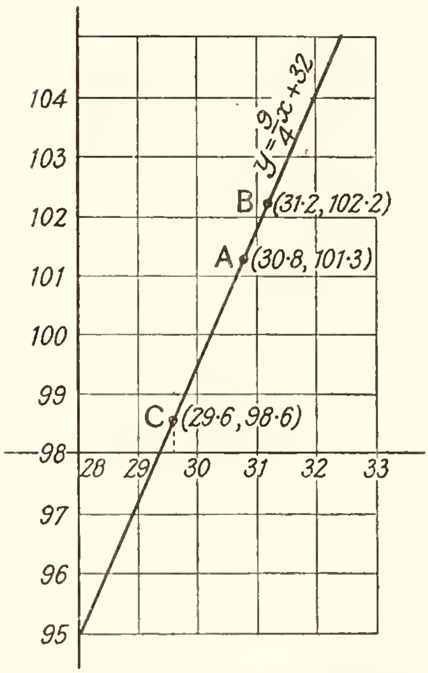

FIG. 40.—Graph $y = \frac{9}{4}x + 32$ showing relation between the Fahrenheit and Réaumur Scales of Temperature.

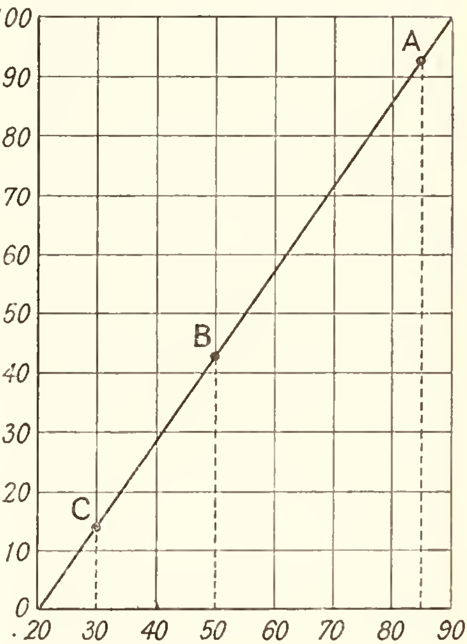

FIG. 41.

(4) Two examiners have marked papers as follows: The first one gave no marks to the worst candidate and 100 marks to the best. The second examiner gave 20 marks to the worst candidate and 90 marks to the best (for the same papers). Draw a graph which will enable you to convert the marks of the two examiners into one another, and use it for converting 85, 50 and 30 marks given by the second examiner into the corresponding marks on the system adopted by the first examiner.

The graph is shown in fig. 41, where it is seen that the marks corresponding to 85, 50 and 30 are 93, 43 and 14, respectively.

(5) The black dots in fig. 42 represent the lengths in cm. (y axis) of a spiral spring fixed at its upper end and stretched by different weights in grammes (x axis) in succession. What criticism can you offer on these observations?

The graph (fig. 42) shows that the 1st, 2nd, 4th and 5th points lie on a perfectly straight line, therefore the law connecting the variables is a linear one. The 3rd point is situated just above the line. Therefore either the reading of the length is too great and should be 46·8 instead of 47·2 cm., or the weight was faulty and should have been 420 instead of 400 grammes. As the second alternative is practically ruled out, we must say that the length 47·2 was too great.

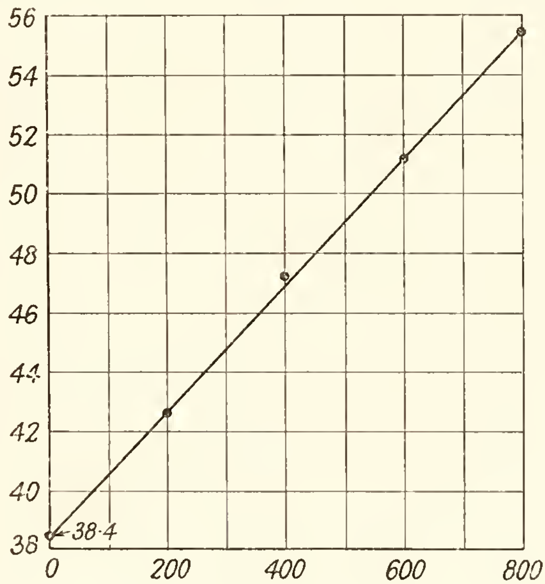

FIG. 42.

The Equation of a Circle (see fig. 43).—(a) If we take the centre of the circle as the origin, then, since for any point P(x, y) on the circumference

$$x^2+y^2 = r^2 \text{ (where } r = \text{length of radius)},$$

$\therefore$ the equation $x^2+y^2 = r^2$ represents a circle whose centre is at the origin.

(b) If the centre is not at the origin, but at a point, say (6, 8), then since for any point P(x, y) (see fig. 44) we have

$$\begin{aligned} \text{OP}^2 &= \text{OM}^2+\text{MP}^2 = (\text{KM}-\text{KO})^2+(\text{NP}-\text{NM})^2 \\ &= (x-6)^2+(y-8)^2; \end{aligned}$$

$\therefore$ if OP $= r$, the equation becomes

$$(x-6)^2+(y-8)^2 = r^2;$$

and generally, if the centre is at the point (h, k), the equation of the circle is

$$(x-h)^2+(y-k)^2 = r^2,$$

which is the general equation of a circle.

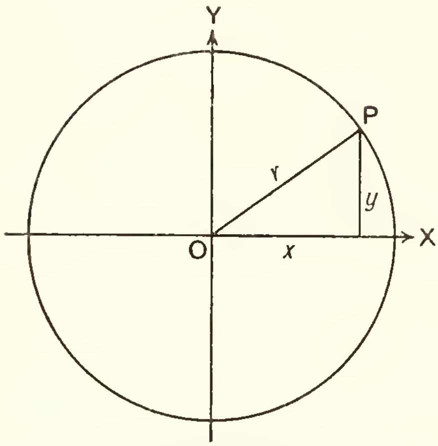

FIG. 43.—The Equation of a Circle of which the Centre is the Origin is $x^2+y^2 = r^2$.

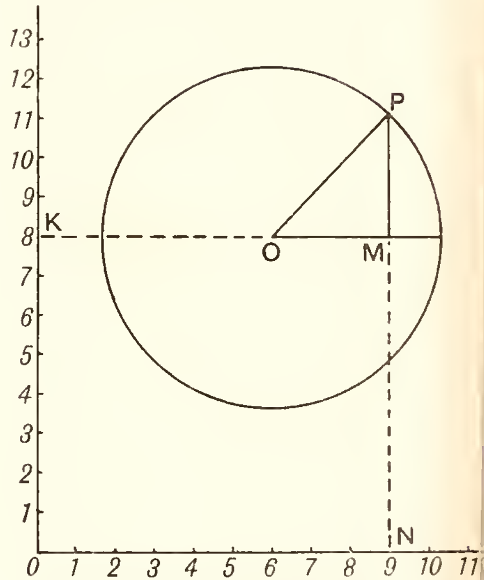

FIG. 44.—General Equation of a Circle $(x-h)^2+(y-k)^2 = r^2$.

Expanding the equation we get

$$x^2+y^2-2xh-2yk+h^2+k^2-r^2 = 0,$$

and since h, k and r are constants we can put $h^2+k^2-r^2 = c$;

$\therefore$ equation of a circle is

$$x^2+y^2-2xh-2yk+c = 0.$$

The Graph of the Function $y = x^2$ (or $\sqrt{y} = \pm x$).

By putting	$x = 0$, y	becomes	$= 0$,	$\therefore$	curve passes through	origin
,,	$x = 1$, y	,,	$= 1$,	$\therefore$	,, ,,	$(1, 1)$
,,	$x = 2$, y	,,	$= 4$,	$\therefore$	,, ,,	$(2, 4)$
,,	$x = 3$, y	,,	$= 9$,	$\therefore$	,, ,,	$(3, 9)$
,,	$x = -1$, y	,,	$= 1$,	$\therefore$	,, ,,	$(-1, 1)$
,,	$x = -2$, y	,,	$= 4$,	$\therefore$	,, ,,	$(-2, 4)$
,,	$x = -3$, y	,,	$= 9$,	$\therefore$	,, ,,	$(-3, 9)$

and so on.

The graph is a parabola passing through the origin, and whose axis is the axis of y (see fig. 45).

Similarly, the graph of the function $y^2 = x$ will be represented by a parabola passing through the origin, and whose axis is the axis of x (see fig. 46).

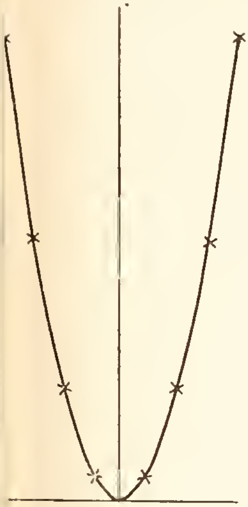

Fig. 45.—Graph of Function $y = x^2$.

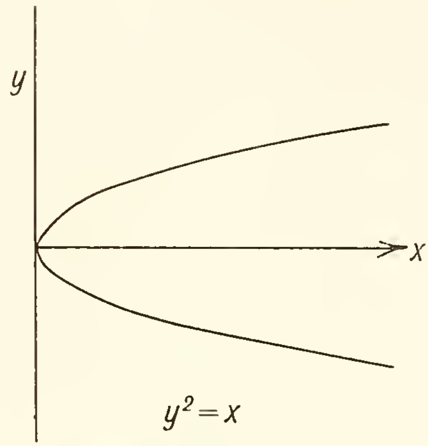

Fig. 46.—Graph of Function $y^2 = x$.

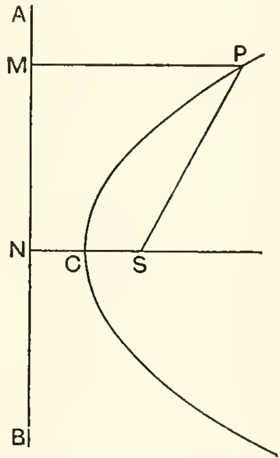

Fig. 47.—To Illustrate Property of a Parabola.

The Property of a Parabola (fig. 47).—The distinguishing character of a parabola is that any point on it is always the same distance from a given fixed line AB as from a given fixed point S.

Thus
$$PS = PM,$$
$$CS = CN, \text{ etc.}$$

The fixed point S is called the *focus*, and the fixed line AB is called the *directrix*.

Consider now such a function as $(y-2) = 3(x+1)^2$.

When $x =$	0,	$y-2 = 3\times 1^2$	$= 3$.	$\therefore$	$y = 5$
,, $x =$	1,	$y-2 = 3\times 2^2$	$= 12$.	$\therefore$	$y = 14$
,, $x =$	2,	$y-2 = 3\times 3^2$	$= 27$.	$\therefore$	$y = 29$
,, $x =$	3,	$y-2 = 3\times 4^2$	$= 48$.	$\therefore$	$y = 50$
,, $x =$	-1,	$y-2 = 3\times 0$	$= 0$.	$\therefore$	$y = 2$
,, $x =$	-2,	$y-2 = 3\times(-1)^2$	$= 3$.	$\therefore$	$y = 5$

and so on.

In this case in order to make the graph of convenient size it is best to choose our scale of representation in such a way that

one scale division = 5 units along Oy and one scale division = unity along Ox.

The graph is shown in fig. 48.

It is a parabola whose vertex is at the point $(-1, 2)$ and whose axis is the line $x = -1$.

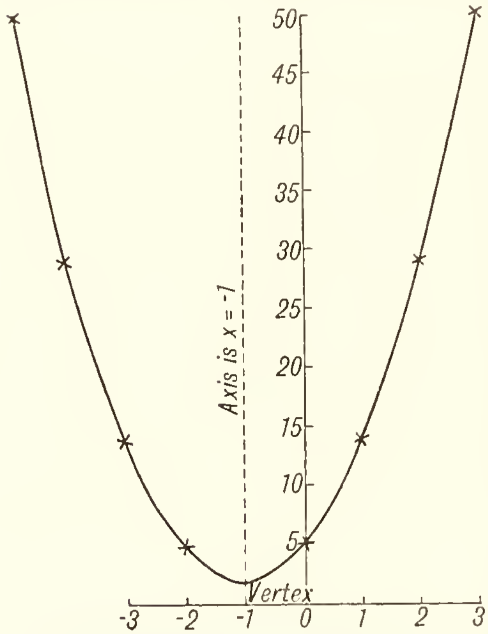

FIG. 48.—Graph of Function $(y-2) = 3(x+1)^2$.

In general, an equation of the form $y = ax^2 + bx + c$ or $ay^2 + by + c = x$, in which one of the variables is of the first degree and the other of the second degree, may be represented graphically by some sort of a parabola.

Examples of Parabolic Functions are:

(i) The law of motion at uniform acceleration (a), which states that the distance covered (s) is proportional to the square of the time (t) $\left(s = ut + \frac{t^2a}{2}\right)$ (u is initial velocity).

(ii) The movement of a projectile in the air when thrown at an angle with the vertical.

(iii) (a) The relation between the side of a square and its area, $y = x^2$; (β) the relation between the radius of a circle and its area, $y = \pi r^2$.

(iv) Schütz-Borissoff law (see fig. 131, p. 356).

(v) The relation between total body length and age during the fœtal period, $T = \left(\frac{L}{28} + 1{\cdot}25\right)^2 + 0{\cdot}74$, where T = age in months and L = length in cm. (see p. 386).

The Ellipse (fig. 49).—The ellipse is a curve possessing the following distinguishing properties:—

(1) The sum of the distances of any point on the curve from two fixed points F, F_1, called the *foci*, is constant; *e.g.* $FP + F_1P = FQ + F_1Q =$ constant.

(2) The distance of any point P on the curve from one of the foci always bears a constant proportion (less than 1) to the distance of the same point from a corresponding fixed line MN called the *directrix*, *e.g.* $\frac{FP}{PM} = \frac{FA}{AK} = e(<1)$. This fixed ratio e is called the *eccentricity* of the ellipse, and, whilst it is

constant for the same ellipse, it has different values for different ellipses.

The diameter AA_1, on which the foci are situated, is called the *major axis*, and is taken as equal to $2a$.

The diameter BB_1, which bisects the major axis at right angles, is called the *minor axis*, and is taken as equal to $2b$.

The first property of an ellipse suggests a *method of describing the curve.* Take a fine thread FPF_1 equal in length to the major axis AA_1. Fix the two ends of the thread to two pins stuck at F and F_1, and by moving a fine-pointed pencil in tight contact with the thread, the half $APBQA_1$ of the curve will

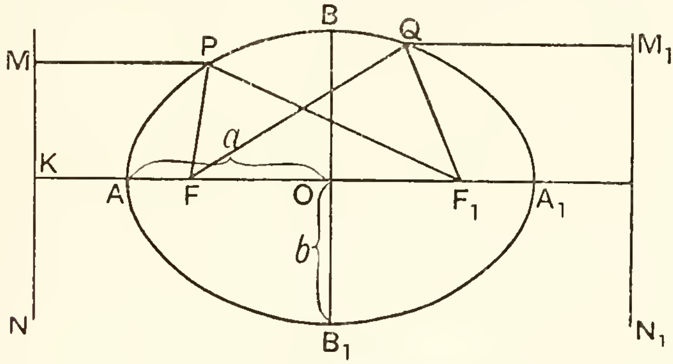

FIG. 49.—Illustrating the Properties of an Ellipse.

be traced out, since in any position such as P, $FP + F_1P$ = length of thread = AA_1. By transferring the thread to the other side of the pins the lower half will be traced out.

Equation of Ellipse.—If we call the major axis $2a$ and the minor axis $2b$, and if we take these axes as the co-ordinate axes, O being the origin, then the equation of an ellipse can be shown to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1. \quad \begin{pmatrix} a = \text{semi-major axis,} \\ b = \text{semi-minor axis.} \end{pmatrix}$$

The eccentricity $e = \sqrt{1-\frac{b^2}{a^2}}$. If $a = b$, *i.e.* when the eccentricity is zero, the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{a^2} = 1,$$

or

$$x^2 + y^2 = a^2,$$

which is the equation of a circle of radius a.

The equation of the ellipse is important in connection with:

(i) The mensuration of the gravid uterus (see p. 311).

(ii) The lines of growth of certain bivalve molluscs (see D'Arcy W. Thompson's "*Growth and Form*," p. 583).

(iii) Elliptical muscles (see S. Haughton's "*Animal Mechanics*").

In astronomy the ellipse is of course of the utmost importance because the orbits of all the planets are ellipses.

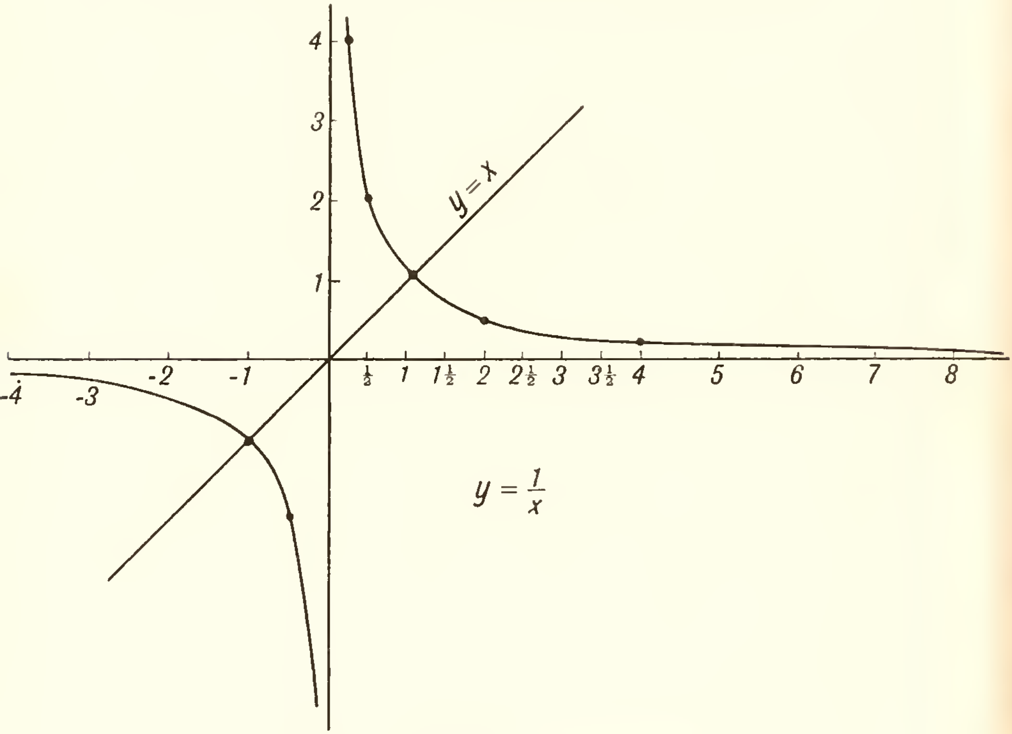

FIG. 50.—Graph of the Function $y = \frac{1}{x}$.

The Graph of the Function $y = \frac{1}{x}$.

By putting	$x = 0,$	y becomes	=	∞
,,	$x = -4,$	y	,,	=	$-\frac{1}{4}$
,,	$x = -2,$	y	,,	=	$-\frac{1}{2}$
,,	$x = -1,$	y	,,	=	-1
,,	$x = 1,$	y	,,	=	1
,,	$x = 2,$	y	,,	=	$\frac{1}{2}$
,,	$x = 4,$	y	,,	=	$\frac{1}{4}$

The resulting graph is given in fig. 50. It is what is called a **rectangular hyperbola.** It will be noticed that this curve possesses the following properties:—

(1) It consists of two symmetrical halves situated in the first and third quadrants of the axes of co-ordinates.

(2) Each half is itself symmetrical with reference to the line $y = x$, which is the axis of the curve.

(3) Although the limbs of the curve approach nearer and nearer to the axes of y and x, they never actually touch these axes. Hence the y and x axes are what are called the *asymptotes* of the curve.

We see, therefore, that when two variables are so related that one varies as the reciprocal of the other, the graph of the function is a rectangular hyperbola.

Examples of Hyperbolic Functions are:

(i) Boyle's law of the relationship between the pressure and volume of a gas at constant temperature, which states that the volume of a gas at constant temperature is inversely proportional to the pressure to which the gas is subjected.

(ii) The extent of ionic dissociation of an electrolyte is inversely proportional to the concentration.

(iii) The dissociation curve of hæmoglobin (*i.e.* the relation of oxygen pressure to oxy- and total hæmoglobin) is a rectangular hyperbola.

All functions of this nature can, as we have seen, be transformed into linear functions by means of logarithms (see p. 115).

EXERCISE.

Plot the curve from the following data and show that it is an hyperbola.

x	0	1	2	3	4	5	6	7	8	9
y	∞	9	4·5	3	2·25	1·8	1·5	1·3	1·13	1

[xy = constant = 9, $\therefore$ curve is a rectangular hyperbola.]

The Graph of the Function $y = x^3 - 12x$.

When $x = 0$, $y = 0$, $\therefore$ curve passes through origin.
,, $x = 1$, $y = -11$
,, $x = 2$, $y = -16$
,, $x = 3$, $y = -9$
,, $x = 4$, $y = 16$
,, $x = -1$, $y = 11$
,, $x = -2$, $y = 16$
,, $x = -3$, $y = 9$
,, $x = -4$, $y = -16$

The graph is shown in fig. 51.

A better way of drawing the graph would be to take different scales of representation for x and y. Thus, if one scale unit of x is made equal to 5 scale units of y, we get a much more detailed curve, as shown in fig. 52.

It will be found that **the graphs of all equations of the form** $y = ax^3 + bx^2 + cx + d$ **have the shape of the letter S.**

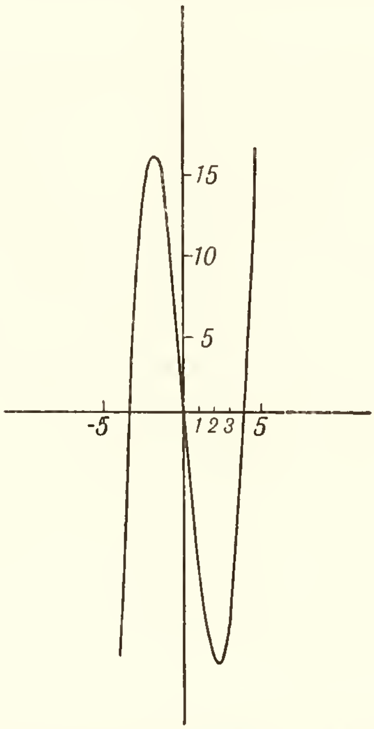

FIG. 51.—Graph of Function $y = x^3 - 12x$.

The Graph of the Function $y = c^x$ (fig. 53).—If we write the equation in the form $2{\cdot}3 \log_{10} y = x$, we can arrange the plotting table as follows:—

y	0	1	2	3	4	5	6	7	8	9	10
$x = 2{\cdot}3 \log_{10} y$	$-\infty$	0	0·69	1·1	1·38	1·61	1·8	1·93	2·08	2·2	2·3

The Graph of $y = e^{-x^2}$ (fig. 54).—This is a most important curve in mathematical statistics (see p. 408 *et seq.*).

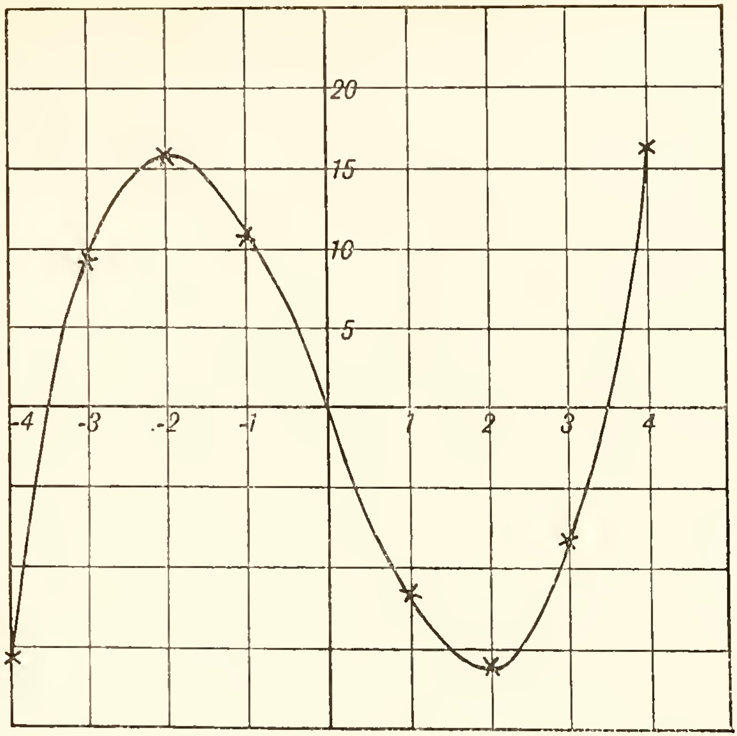

Fig. 52.—Graph, $y = x^3 - 12x$, drawn with Different Scales of Representation for x and y. It crosses the x axis at the three values of x for which $x^3 - 12x = 0$, viz. the origin and the points $x = \pm\sqrt{12} = \pm 3{\cdot}464$.

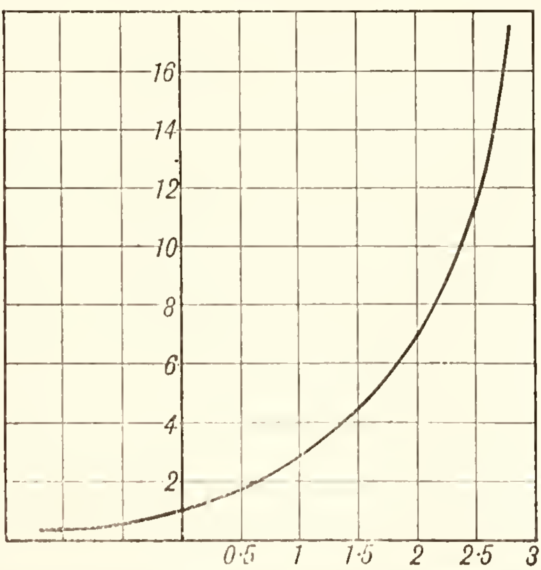

Fig. 53.—Graph of $y = e^x$ (Compound Interest Law).

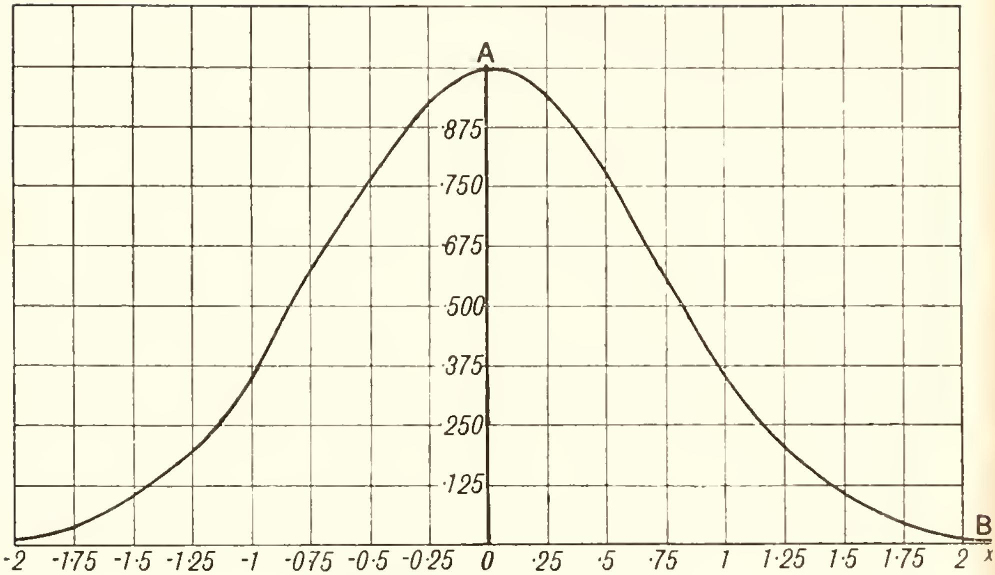

FIG. 54.—Graph of $y = e^{-x^2}$.

If we put $2{\cdot}3 \log_{10} y = -x^2$, or $x^2 = -2{\cdot}3 \log_{10} y$, we get the following plotting table (since for values of y less than 1, the value of $-2{\cdot}3 \log_{10} y$ is + ve, and therefore $\sqrt{-2{\cdot}3 \log_{10} y}$ is real):—

y	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
$x = \pm\sqrt{-2{\cdot}3 \log_{10} y}$	$\pm\infty$	$\pm 1{\cdot}52$	$\pm 1{\cdot}27$	$\pm 1{\cdot}10$	$\pm 0{\cdot}96$	$\pm 0{\cdot}83$	$\pm 0{\cdot}71$	$\pm 0{\cdot}60$	$\pm 0{\cdot}47$	$\pm 0{\cdot}32$	0·0

The Graphs of Circular Functions.

$$y = \sin x.$$

When $x = 0°$, $y = 0$
,, $x = 30°$, $y = \frac{1}{2}$
,, $x = 60°$, $y = \frac{\sqrt{3}}{2}$
,, $x = 90°$, $y = 1$
,, $x = 120°$, $y = \frac{\sqrt{3}}{2}$
,, $x = 150°$, $y = \frac{1}{2}$
,, $x = 180°$, $y = 0$

When $x = 210°$, $y = -\frac{1}{2}$
,, $x = 240°$, $y = -\frac{\sqrt{3}}{2}$
,, $x = 270°$, $y = -1$
,, $x = 300°$, $y = -\frac{\sqrt{3}}{2}$
,, $x = 330°$, $y = -\frac{1}{2}$
,, $x = 360°$, $y = 0$
etc.

The graph is shown in fig. 55.

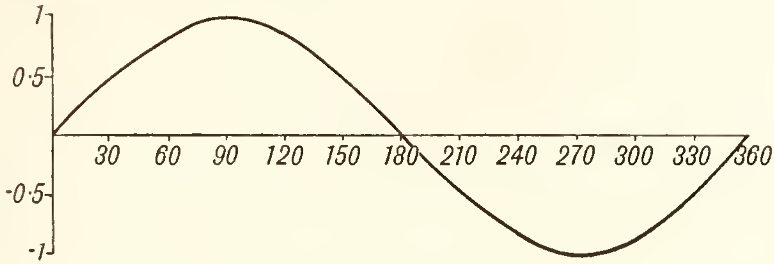

FIG. 55.—Graph of $y = \sin x$.

In a similar way $y = \cos x$ (see fig. 56) and $y = \tan x$ may be represented by graphs.

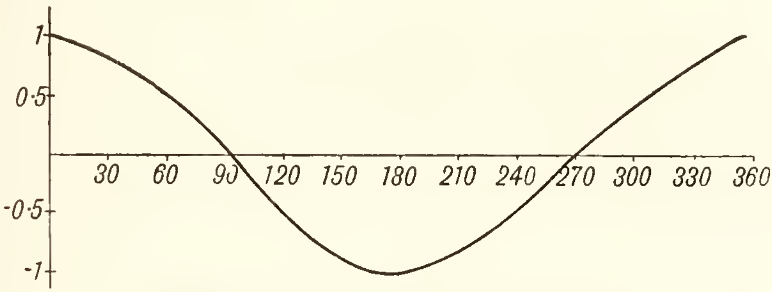

FIG. 56.—Graph of Function $y = \cos x$.

Graphical Method of Solving Equations.—In all the graphs we have plotted the ordinates of course become zero at the points at which the graph of the particular function happens to cut the abscissa. Hence, if we read off the values of x at those points of intersection we obtain the values of x which make the function, $f(x)$, vanish, *i.e.* we obtain the roots of the equation $f(x) = 0$. **Graphs, therefore, afford a convenient method of solving those numerous equations containing one or two unknowns of higher degrees than the second that defy the ordinary algebraical methods.** Thus the graph of fig. 52 cuts the abscissa at the origin (*i.e.* where $x = 0$) and also at the points where $x = +3·5$ and $-3·5$, approximately. Hence the roots of the equation $x^3 - 12x = 0$ are $x = 0$ or $\pm 3·5$ (approximately).

In order to find more accurately the values of the two roots approx. $\pm 3·5$, we must plot the points near $\pm 3·5$ as follows:—

For $x = \pm 3·4, \quad y = \pm (3·4)^3 \mp 12 \times 3·4 = \mp 1·50$

$x = \pm 3·5, \quad y = \pm (3·5)^3 \mp 12 \times 3·5 = \pm 0·875.$

Therefore between these two values of x, y has changed from negative to positive, or *vice versa*, *i.e.* it must have passed through zero. Therefore the value of x which makes the function equal to zero must be numerically greater than $\pm$ 3·4 and less than $\pm$ 3·5. By plotting the points near $x = \pm 3{\cdot}45$, one can get more accurate values of these roots, which will be found to be $\pm$ 3·46 to two decimal places.

Note.—As a matter of fact, this particular equation could be solved more expeditiously by factors. Thus $x(x^2 - 12) = 0$, $\therefore$ $x = 0$, or $\pm 2\sqrt{3}$. It has merely been taken as an illustration.

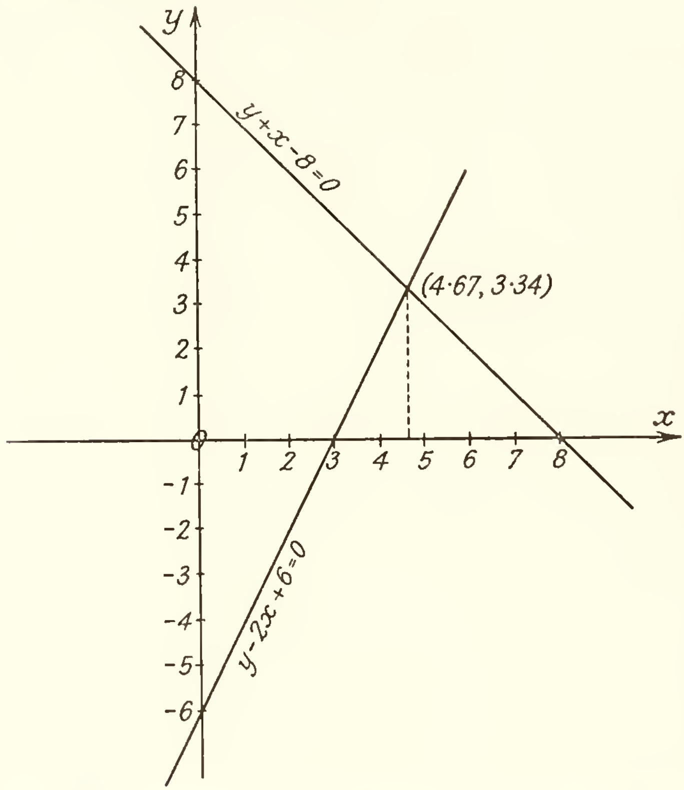

FIG. 57.—Illustrating Graphic Solution of Simultaneous Equations.

Similarly, **simultaneous equations of two unknowns** may sometimes be solved graphically by drawing the graph for each of the two equations. The point of intersection of the two graphs gives the roots required, because at this point the functions have common values of x and y.

Thus the graphs of $y + x - 8 = 0$ and $y - 2x + 6 = 0$ intersect at the point (4·67, 3·34) as can be seen approximately from fig. 57

and therefore the solution of these two simultaneous equations is $x = 4{\cdot}67$, $y = 3{\cdot}34$.

"Two-graph" Method of Solving Equations containing One Unknown.—In the case of equations containing high powers of x, or some other functions (such as trigonometric or logarithmic functions, etc.) of x, it is sometimes simpler to solve them by means of two graphs instead of one, especially if we can make one of the graphs a straight line. This will be clear from the following example. Supposing we have to solve $x^3 - 10x = 7$. This is the same as $x^3 = 10x + 7$.

Equate each side of the equation to y and we have $y = x^3$, $y = 10x + 7$. On drawing these two separate graphs (which is much easier than drawing the single graph of $y = x^3 - 10x - 7$) the abscissæ of the points of intersection are the roots of the original equation, inasmuch as they satisfy each of the equations (see fig. 58). They are seen to be about $3{\cdot}47$, $-0{\cdot}75$, $-2{\cdot}72$.

The advantage of this method is that once one has obtained the curve of $y = x^3$, one can solve any other cubical equation by means of it, *provided the equation contains no x^2 term.* Thus to solve $x^3 - 2x - 5 = 0$, one has to find the points at which the line $y = 2x + 5$ intersects the graph $y = x^3$. It will be seen (fig. 58) that it intersects the curve at one point only, at approximately $x = 2{\cdot}1$, which is one root. The other two roots are therefore imaginary.

The curve $y = x^3$ is called a cubical parabola.

Cardan's Form.—*Cases in which the equation contains a term of the second degree in x, e.g.* $3x^2$ or $4x^2$ or ax^2, in addition to the other terms, must be solved either by plotting the graph of the function and finding where it intersects the x axis, or by means of a **special device** introduced **for eliminating the x^2 term.** Thus, supposing we have $x^3 - 3x^2 - 10x + 24 = 0$. By proceeding in the ordinary way the graph will be found to cut the x axis at the points $x = -3$, $x = 2$ and $x = 4$, which are therefore the roots of the equation. This method, however, is troublesome, and we therefore eliminate the x^2 term as follows:—

Put $x = (x_1 + a)$, and the equation becomes

$$(x_1 + a)^3 - 3(x_1 + a)^2 - 10(x_1 + a) + 24 = 0,$$

i.e. $$(x_1^3 + 3x_1^2 a + 3x_1 a^2 + a^3) - 3(x_1^2 + 2x_1 a + a^2) - 10(x_1 + a) + 24 = 0,$$

i.e. $$x_1^3 + 3x_1^2(a - 1) + x_1(3a^2 - 6a - 10) + (a^3 - 3a^2 - 10a + 24) = 0.$$

Hence if this expression is not to contain x_1^2, we must have $(a - 1) = 0$ or $a = 1$, and then the equation becomes

$$x_1^3 + x_1(3 - 6 - 10) + (1 - 3 - 10 + 24) = 0,$$

i.e. $x_1^3 - 13x_1 + 12 = 0$, and this is the same equation as $x^3 - 3x^2 - 10x + 24 = 0$ where $x_1 = (x - 1)$.

Hence if we find the points of intersection of the line $y = 13x_1 - 12$ with the graph $y = x_1^3$ (for which of course the graph $y = x^3$ will serve),

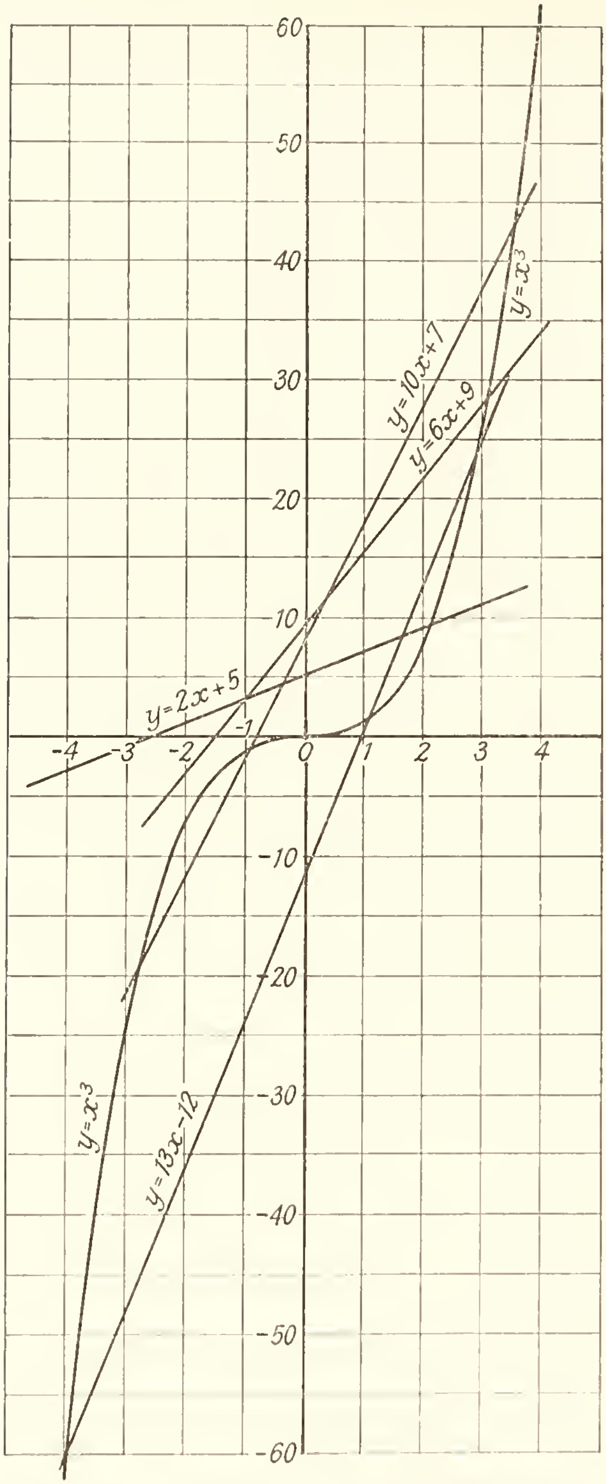

FIG. 58.—Graphic Solution of Cubic Equations.

we get the three values of x_1 which are the roots of $x_1^3 - 13x_1 + 12 = 0$, and by adding 1 to each of these values we get the roots of the original equation $x^3 - 3x^2 - 10x + 24 = 0$ (because $x = x_1 + 1$).

Thus the abscissæ of the points of intersection of $y = 13x_1 - 12$ with $y = x_1^3$ are $x_1 = 1$, $x_1 = 3$ and $x_1 = -4$ (see fig. 58). Therefore the roots of the original equation are 2, 4 and -3.

The form of a cubic equation containing only three terms and in which the x^2 is missing (*i.e.* $x^3 + ax + b = 0$) is called **Cardan's form.** *Every cubic equation can be reduced to this form* by the artifice just explained. If the original equation is of the form $Ax^3 + Bx^2 + Cx + D = 0$, it must first be divided throughout by A to make the coefficient of x^3 unity.

The solution of quadratic equations ($x^2 + ax + b = 0$) by means of two graphs, $y = x^2$ and $y = -ax - b$, will present no difficulty to the reader.

Equations of higher degree than the third, if they happen to be of the type $x^n + ax + b = 0$ or are reducible to this form, can also be solved by the two-graph method. Usually, however, they are not so reducible, and must therefore be solved by plotting the whole function and finding its points of intersection with the x axis.

EXAMPLES.

(1) Solve $x^2 - 3x + 1 = 0$.

Draw the graph in the usual way. It will be found to cut the x axis at points where $x = 0{\cdot}4$ and $2{\cdot}6$ very nearly. If, now, the points in the neighbourhood of $x = 0{\cdot}4$ and $x = 2{\cdot}6$ be plotted on a large scale, it will be seen that the curve cuts the x axis very close to $x = 0{\cdot}38$ and $x = 2{\cdot}62$ (see further, Example (3), p. 313). The process can be repeated in order to get closer and closer approximations.

Alternate Method.—Plot $y = x^2$ and $y = 3x - 1$ and verify that the two graphs intersect at the points whose abscissæ are $0{\cdot}38$ and $2{\cdot}62$ approximately.

(2) Solve the equation $x^3 - 6x^2 + 6x - 5 = 0$.

Converting this into Cardan's form, it will be found to be

$$x_1^3 - 6x_1 - 9 = 0, \quad \text{where} \quad x_1 = (x-2).$$

The line $y = 6x_1 + 9$ will be found to cut the graph $y = x_1^3$ at the point whose abscissa is $x_1 = 3$. Therefore $x = 5$ (see fig. 58). As the line does not cut the cubical parabola more than once, the other roots are imaginary.

(3) A cylindrical vessel of radius 2 feet and with its axis horizontal has to be filled with water to one quarter of its full capacity. To what level will the water reach?

Let fig. 18 (p. 54) represent a section of the vessel. We have to find the level of AB, *i.e.* the height MC, when the segment AMB is equal to 1/4 of the area of the circle, viz. $\pi r^2/4$.

$$\text{Hence} \qquad \frac{r^2}{2}(\theta - \sin\theta) = \frac{\pi r^2}{4}, \quad \text{whence} \quad \theta - \frac{\pi}{2} = \sin\theta.$$

This equation, involving as it does an angle and one of its trigonometrical

functions, can only be solved graphically, and the two-graph method is the best.

In fig. 59 the y axis is graduated in radians—each unit division corresponding to half a radian—and the x axis is graduated in degrees—each unit division corresponding to 15°.

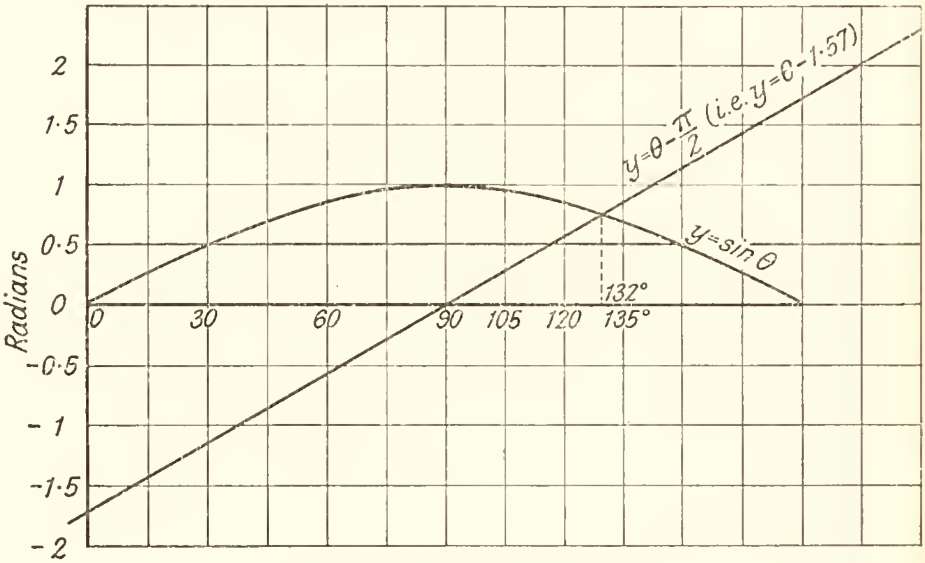

FIG. 59.—Graphs for $\theta - \frac{\pi}{2} = \sin \theta$.

It is seen that the line $y = \theta - \frac{\pi}{2}$ (or $y = \theta - 1{\cdot}57$) intersects the graph $y = \sin \theta$ at a point whose abscissa is $\theta = 132°$ approximately. Having thus found θ, we can compute the value of OC (fig. 18) since

$$OC = r \cos \frac{\theta}{2} = r \cos 66° = 0{\cdot}41r.$$

$$\therefore \quad \text{Depth of Water} = CM = r - 0{\cdot}41r = 0{\cdot}59r.$$

But $\qquad r = 2$ feet, $\quad \therefore \quad CM = 0{\cdot}59 \times 2 = 1{\cdot}18$ feet.

CHAPTER IX.

NOMOGRAPHY.

All the graphs that have been given in the last chapter represent the relationship between two variables only, and we have seen how such graphs enable us to find the value of the dependent variable (y) for any value of the independent variable (x). But how are we to deal with a mathematical formula connecting three, or even more, variables? For example, we know that if W = the weight of a person in kilograms and H = his height in centimetres, then the surface area of his body, S, in square centimetres, is expressed by the Du Bois formula $S = 71{\cdot}84W^{0.425}H^{0.725}$, which is an equation connecting *three* variables.

Again, we know that if C = number of Calories produced by 1 square metre of body surface of a person per hour, then the number of Calories (T) produced by the person in 24 hours is given by

$$\begin{aligned} T &= 24SC \text{ (where S = surface area of body)} \\ &= 24 \times 71{\cdot}84CW^{0.425}H^{0.725}, \end{aligned}$$

which, of course, is an equation connecting *four* variables.

Further, problems connected with the mechanism of gaseous interchange in the blood, as studied by L. J. Henderson, may have to deal with as many as seven or more variables, such as (1) the ratio $\frac{[A_c]}{[A_s]}$, where $[A_c]$ and $[A_s]$ are the concentrations of anions in the red cells and serum, respectively; (2) the volume of the red cells; (3) the total CO_2 content of the blood; (4) the CO_2 tension; (5) the H-ion concentration of the serum; (6) the oxygen tension; (7) the percentage saturation of hæmoglobin with oxygen.

It is obvious that such a number of variables cannot be represented on a *plane* surface by means of Cartesian coordinates as described in the last chapter for the case of two variables. It is equally obvious that the greater the number of variables, and the more complex the formula or law con-

necting them, the more desirable it is to have some ready means whereby the value of the dependent variable can be ascertained at a glance from a knowledge of the values of the other variables. Thus, taking the Du Bois formula again, $S = 71 \cdot 84 W^{0.425} H^{0.725}$. If we had to solve this equation in the ordinary mathematical way, to find the surface area (S) of any given individual knowing his weight (W) and his height (H), the labour expended would be great and tedious. Thanks, however, to the genius of d'Ocagne (1884), it is possible to represent any function consisting of any number of variables by means of a number of lines drawn on a *plane* surface (each line representing one variable) and graduated in such a way that the value of the dependent variable can be read off at a glance on the appropriate line from a knowledge of the values of the other variables.

In the case of the Du Bois formula, for instance, it is possible to draw a chart (fig. 60) consisting of three parallel lines situated at fixed distances from one another and graduated in such a way that

(i) If the two external lines are graduated respectively with scales representing W and H, and the middle one is graduated with a scale representing S, then

(ii) any straight line ACB joining any particular graduation A on the W scale with any other graduation B on the H scale will cut the S scale at a point C corresponding in value to $71 \cdot 84 W^{0.425} H^{0.725}$. This line ACB is called the "index line."

Thus the line ACB joining the point 24 on the weight scale with the point 110 on the height scale will cut the surface scale at a point corresponding to 8375, which means that the surface area of a person 24 kilograms in weight and 110 cm. in height is 8375 sq. cm. (For the manner of construction, see p. 146 *et seq.*)

Because such a chart enables us, by means of one diagram, to find the value of any one variable for given values of the other variables in a given "law," therefore the chart is called a **nomogram** (nomos = law). It is also called an **alignment chart,** because the corresponding points lie in one line, *i.e.* are collinear.

The Construction of Nomograms.

The Sum and Difference Nomogram.—(*a*) In order to understand the principles of *nomography* or the construction of nomograms, we shall construct such a chart for a simple law like $x = y \pm z$ connecting three variables. If we draw three parallel lines at equal distances apart, and call the two outside

lines the y and z scales, and the middle line the x scale, and then graduate y *and* z with the *same* unit, say, one scale division is 1 cm. long, and the scale x with *half* the unit, say, one scale division is only 0·5 cm. long (see fig. 61), then if we join any

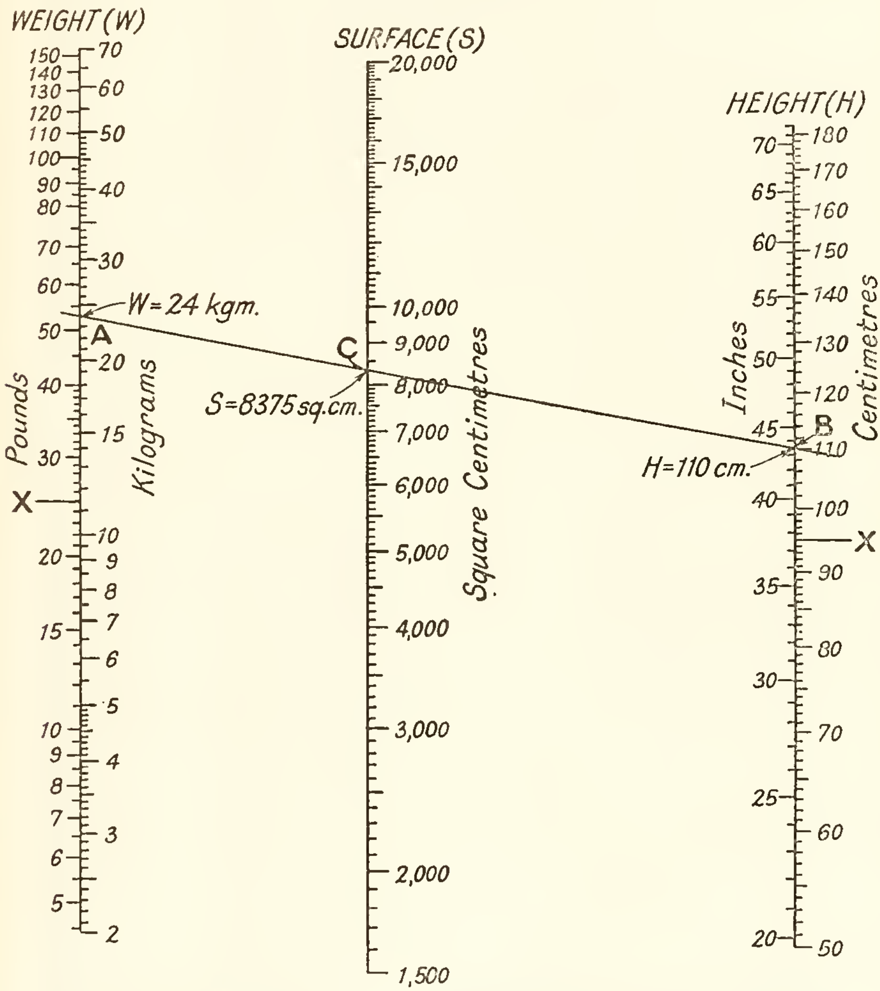

FIG. 60.—The Nomogram for $S = 71 \cdot 84 W^{0 \cdot 425} H^{0 \cdot 725}$ (Du Bois' Law).

value of y with any value of z, the line will cut the x scale at a point corresponding to $y \pm z$.

Thus, in the diagram, the line AB joining the point 3·8 on the y scale with the point 2·4 on the z scale cuts the x scale at the point D corresponding to 6·2 (*i.e.* 3·8 + 2·4), whilst the line A_1B_1 joining the point 2·3 on the y scale with the point $-0·8$ on the z scale cuts the x scale at the point 1·5 (*i.e.* 2·3 − 0·8).

This is so, because, if we look at fig. 61, we shall see at once that by the law of similarity of triangles

$$CD = \tfrac{1}{2}0_1A \quad \text{and} \quad 0_3C = \tfrac{1}{2}0_2B.$$

$$\therefore \quad 0_3D \text{ (which } = 0_3C + CD) = \tfrac{1}{2}(0_2B + 0_1A).$$

Similarly, $$0_3C_1 = \tfrac{1}{2}(0_1A_1 - 0_2B_1).$$

But as we graduated the x scale with *half* the unit of the y and z scales,

$\therefore$ the distance 0_3D *as read off on the scale*

$$= 0_1A + 0_2B,$$

and the distance 0_3C_1 as read off on the scale

$$= 0_1A_1 - 0_2B_1.$$

$$\therefore \quad x = y \pm z.$$

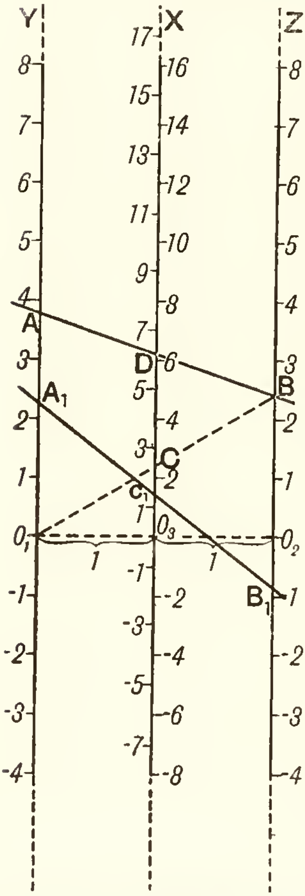

Fig. 61.—Nomogram for $x = y \pm z$.

Nomogram for $x = m_1y \pm m_2z$.—Let us take a specific example. The calorific values of protein, carbohydrate and fat are approximately 4, 4 and 9 calories per gramme respectively. *Construct a nomogram for finding the total number of calories contained in any diet of known composition.*

Since protein and carbohydrate have the same heat value, therefore a diet consisting of P grammes of protein and C grammes of carbohydrate will yield $4(P + C)$ calories. We can, therefore, designate the combined quantity of these two substances, viz. $(P + C)$, by y, and if we call the number of grammes of fat z, and the total number of calories x, we have $x = 4y + 9z$. To construct the requisite nomogram, one can proceed in a number of ways, each of which will give a different chart, which, however, will solve the equation $x = 4y + 9z$, in the same way as graphs drawn to different scales of x and y will look different but will solve the same equation (*cf.* figs. 51 and 52, pp. 124 and 125). The simplest method is the following:—

Place two parallel lines (fig. 62) at a convenient distance apart and call them the y and x scales respectively. As the

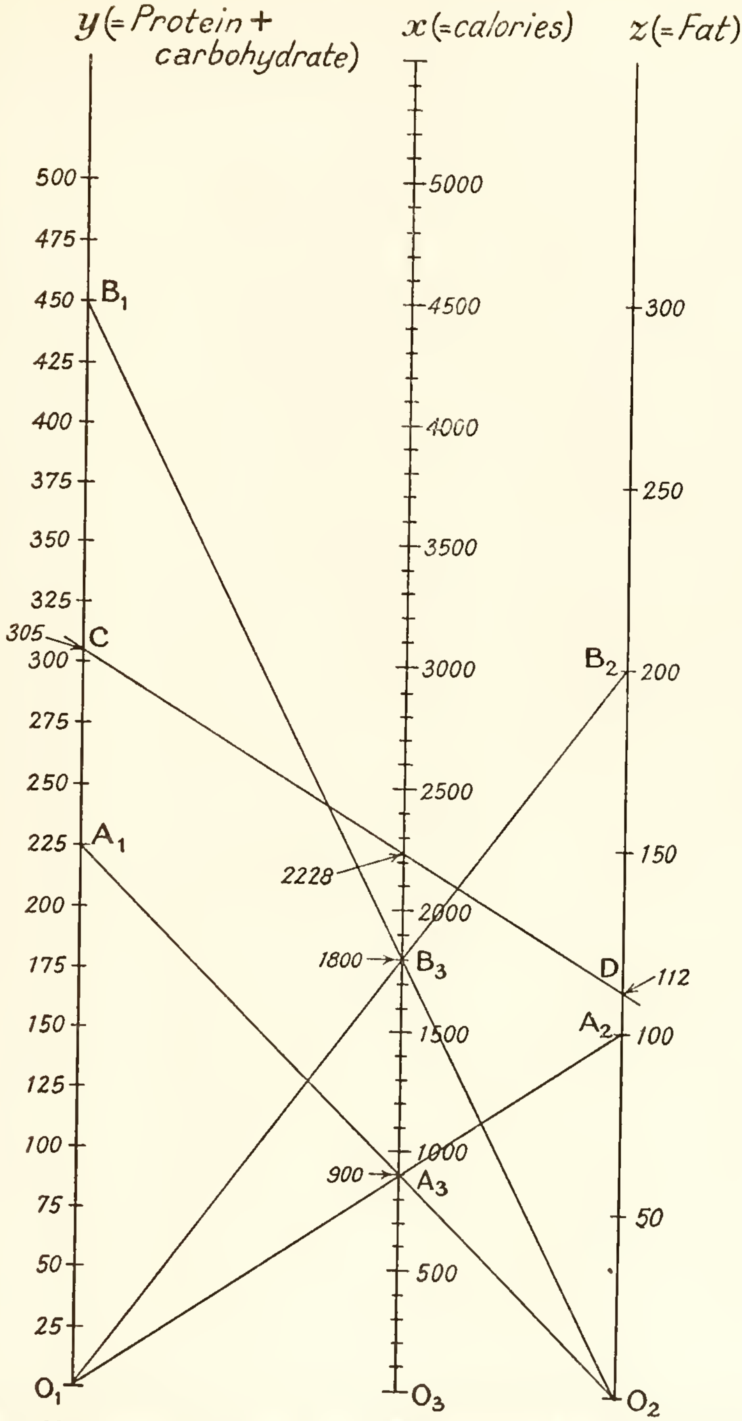

FIG. 62.—Nomogram for $x = 4y + 9z$, to Calculate the Heat Value of a Diet.

combined amount of protein and carbohydrate in a diet is more than that of fat, the y scale will have to be graduated with a smaller unit than the z scale. Let 1 inch of length represent, say, 75 grammes on the y scale and only 50 grammes on the z scale, *i.e.* make the graduation units of y and z in the proportion of 2 : 3.

We now have to find the position of the x scale and the magnitude of its graduation, which is done *automatically*, as follows: Take any graduation on the y scale, say, the point marked 225 (indicated on the chart by A_1). Since 225 grammes of protein and carbohydrate yield $225 \times 4 = 900$ calories, therefore if we join A_1 with O_2 (*i.e.* no fat), the number 900 must lie *somewhere* on the line A_1O_2. Take the point A_2 on the z scale, which represents 100 grammes of fat, which quantity also yields 900 calories, and join it with O_1 (*i.e.* no protein and carbohydrate); the number 900 will then also lie *somewhere* on the line A_2O_1. It follows that the point A_3, at which the lines A_1O_2 and A_2O_1 intersect, must represent the position of the graduation point 900 on the x or calories scale. Similarly the lines B_1O_2 and B_2O_1 will intersect at the point B_3 representing 1800 calories on the x scale. Join A_3B_3, and it will be found that A_3B_3 is parallel to the y and z scales; also the portion A_3B_3 is equal to 900 ($= 1800 - 900$) calories. Continue A_3B_3 up and down in this way and the position of the x scale as well as its graduation are obtained. The nomogram is now complete. It shows, for example, that a diet containing, say, 305 grammes of protein and carbohydrate (together) and 112 grammes of fat, contains 2228 calories, the line CD through the respective points on the y and z scales cutting the x scale at the point corresponding to 2228. (Arithmetically $305 \times 4 + 112 \times 9 = 2228$.) Incidentally it will be found that the x scale is so situated that (i) it is parallel to the other two scales, (ii) its distance from the y scale is to that from the z scale in the proportion of 3 : 2, *i.e.* in the inverse proportion of the graduation units of the y and z scales.

Fig. 63 shows an alternative nomogram for the same function, in which the graduation units of the y and z scales are in the proportion $m_1 : m_2$ (*i.e.* 4 : 9 in this case) and the x scale is midway between the y and z scales and parallel to them. As the x zero point O_3 is collinear with the zeros O_1 and O_2 of the other scales, once any one graduation A_3 is found automatically the graduation unit of the x scale is determined.

Note.—The reader will realise that it is hardly necessary to go to the trouble of constructing a nomogram for a simple addition formula like

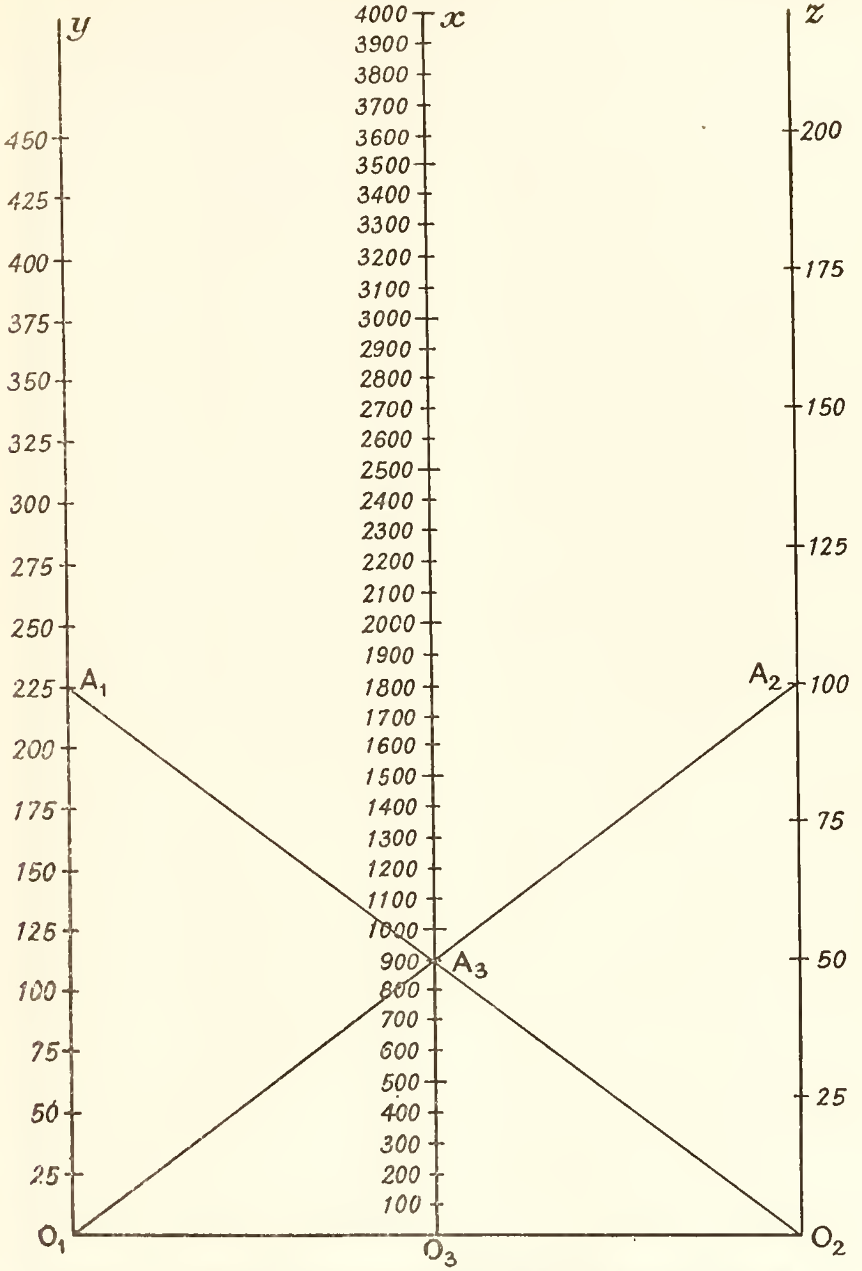

FIG. 63.—Alternative Nomogram for the Expression $x = 4y + 9z$.

this, but it has been worked out in full to illustrate the principle of nomography, since a familiarity with the principles of sum and difference nomograms is necessary for understanding the construction of nomograms for more complicated expressions.

Sum and Difference Nomogram for Four Variables, $x = m_1y \pm m_2z \pm m_3t$.*—It will be clear that if in fig. 64 we first construct a "sum and difference" nomogram for *three* variables, and call the x scale x_1, we get the relationship $x_1 = y \pm z$. The x_1 scale, which is called the *Reference Line* or *Pivotal Line*, because it is merely used as a "pivot" for finding the appropriate

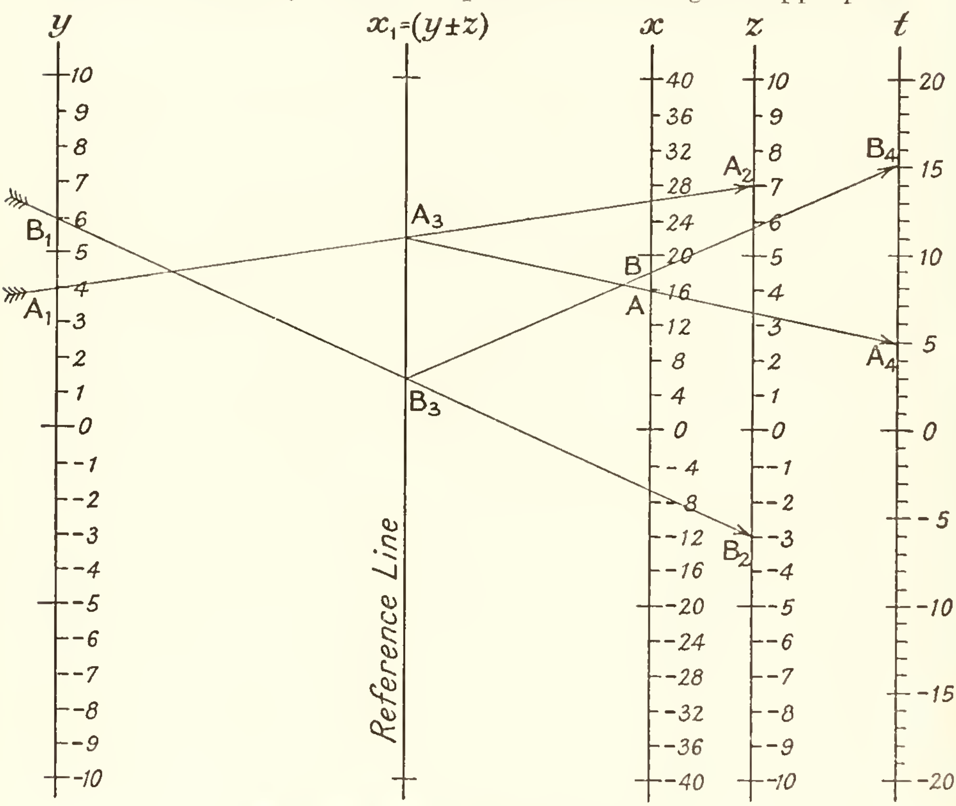

FIG. 64.—Nomogram for $x = y \pm z \pm t$.

point on the x scale, will, as we have seen, be midway between the y and z scales and will have half their graduation unit. If now we take parallel x and t scales and place them in such a way that the x scale is midway between the x_1 and t scales (as in fig. 64), graduating the t scale with the *same* unit as the x_1 scale (*i.e.* with *half* the y and z unit), and also graduate the x scale with *half* the x_1 and t unit (*i.e.* with a *quarter* of the y and z unit), then a line joining any two points on the x_1 and t scales will cut the x scale at a point corresponding to $x = x_1 \pm t$. But as $x_1 = y \pm z$, we have $x = y \pm z \pm t$. The way in which such a nomogram is used is as follows:—

* y, z and t stand for positive numbers.

Take any two given points A_1 and A_2 or B_1 and B_2 on the y and z scales, and find where the line joining them cuts the x_1 scale: this point, A_3 or B_3 in fig. 64, gives the relation $x_1 = y \pm z$. Join this point with the given point A_4 or B_4 respectively on the t scale, and the line thus obtained will cut the x scale at the graduation point A or B respectively, which satisfies the relation $x = y \pm z \pm t$.

E.g., in fig. 64, the line joining graduation 4 on the y scale with graduation 7 on the z scale cuts the reference line x_1 $(= y \pm z)$ at the point A_3, and the line joining A_3 with the point 5 on the t scale intersects the x scale at the point 16, thus giving $x = y+z+t = 4+7+5 = 16$. Similarly the line joining the point 6 on the y scale with the point -3 on the z scale, intersects the reference line x_1 at the point B_3, and the join of B_3 with the point 15 on the t scale cuts the x scale at the point 18, thus giving $x = y-z+t = 6-3+15 = 18$.

Note.—As the x_1 (*i.e.* the $y \pm z$) scale is merely used as a Reference Line for the purpose of finding the point from which to draw the required line to the given point on the t scale, and is not used with the object of ascertaining the value of $x \pm y$, which is not necessary for our purpose, it is obvious that it is unnecessary to graduate the x_1 scale at all. Its *position* with relation to the other lines alone is needed, and its graduation may therefore be omitted.

Sum and Difference Nomogram for n Variables, viz. $x = m_1 y \pm m_2 z \pm m_3 t \pm \ . \ . \ . \pm m_n w$.—By continuing the process just described it is clearly possible to construct a general "Sum and Difference" nomogram for any number of variables. The reader will observe, however, that with each addition of a variable the graduation unit of the x scale has to be halved, so that with several variables the graduation unit of the x scale will soon become so small as to be impracticable. This difficulty is overcome by the *modified method* shown in fig. 65, in which the *z scale is placed midway between the y and x_1 scales* (instead of the x_1 scale midway between the y and z scales) and *the z scale is graduated with half the y unit in the opposite direction* (*i.e.* +ve down; −ve up), keeping the unit of the x_1 scale the same as that of the y scale, but in the opposite direction. The x_1 scale will now give the relation $x_1 = y \pm z$, but with an undiminished unit. For,

$$A_2C = \tfrac{1}{2}A_3B_3,$$

and

$$CB_2 = \tfrac{1}{2}A_1B_1.$$

$$\therefore \quad A_2B_2 = \tfrac{1}{2}(A_1B_1 + A_3B_3).$$

But A_2B_2 being graduated with half the unit of A_1B_1 and

A_3B_3, the distance A_3B_3 as read off on the x_1 scale is equal to $A_1B_1+A_2B_2$, *i.e.* $x_1 = y+z$. (Due allowance for sign must be made here.)

The process is now continued in the same way, *i.e.* the t scale is placed midway between the y and z scales and graduated

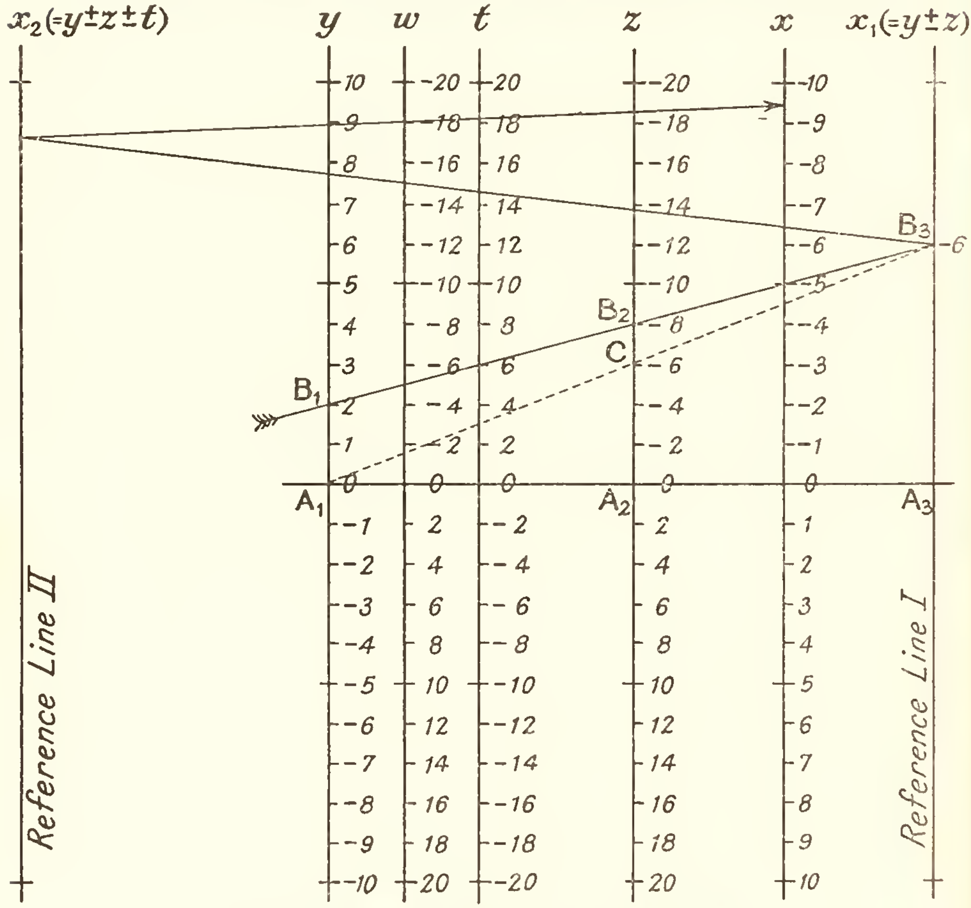

FIG. 65.—Nomogram for $x = y \pm z \pm t \pm w$.

with half the y unit in the *upward* direction (*i.e.* +ve up), and the x_2 scale (reference line II) placed as far to the left of t as the x_1 line is to the right. As there is still another variable, w, the x_2 scale need not be graduated. The w scale is placed midway between the y and t scales and graduated also with half the y unit *downwards* (*i.e.* +ve down, −ve up). Finally the x scale, representing $y \pm z \pm t \pm w$ is placed midway between the z and x_1 scales and graduated with the full y unit *downwards* (*i.e.* +ve down, −ve up).

E.g., the line B_1B_2 meets scale x_1 at B_3, which represents $2-8$ (*i.e.* $y-z$) $= -6$. The join of B_3 with the point $14\frac{1}{2}$ on the t scale meets x_2 at a point which represents $-6+14\frac{1}{2}$

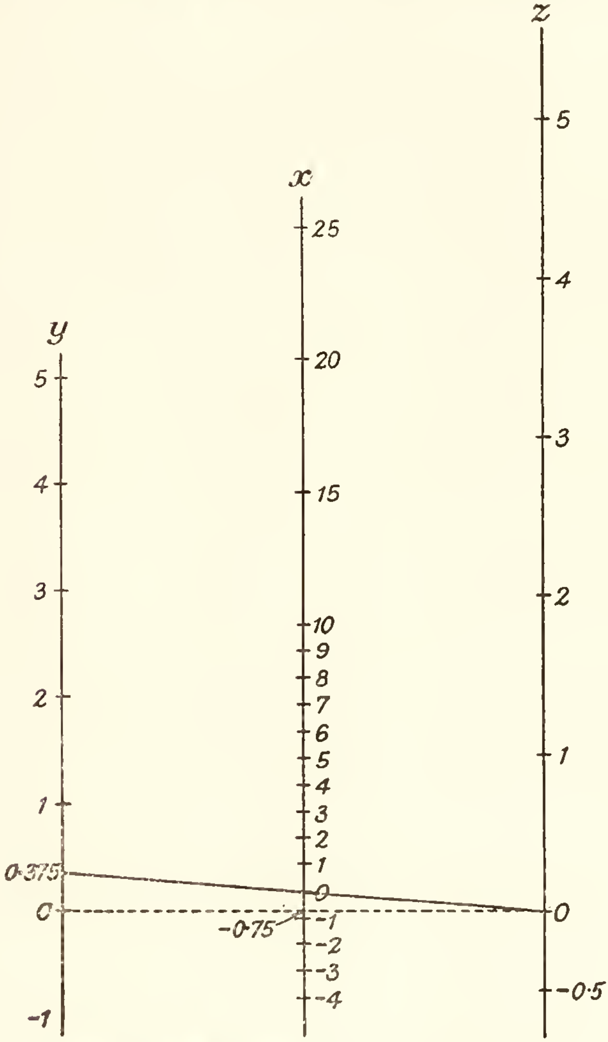

FIG. 66.—Nomogram for $x = 2y+3z-0{\cdot}75$.

(*i.e.* $y-z+t$) $= 8\frac{1}{2}$. The join of this point with the point -18 on the w scale meets the scale x at a point which represents $8\frac{1}{2}-18$ (*i.e.* $y-z+t-w$) $= -9\frac{1}{2}$.

The General Sum and Difference Nomogram for n Variables is $x = m_1y \pm m_2z \pm m_3t \pm \ldots \pm m_nw + c$, where c is a numerical constant, positive or negative (*e.g.* 1, 2, $-1{\cdot}5$, etc.).

In all the nomograms so far considered the zeros of all the

scales have been collinear. If, however, the expression for which a nomogram has to be drawn contains a numerical constant c, the zeros will not be in the same straight line. Fig. 66 shows a nomogram for $x = 2y+3z-0{\cdot}75$. The y and z scales are graduated with units in the proportion of 2 : 3 and the x scale is situated midway between them. The line joining the zeros of the y and z scales must clearly meet the x scale at $x = -0{\cdot}75$ (shown by the dotted line), while the line joining the point 0·375 on the y scale with zero on the z scale must cut the x scale at the point $2\times0{\cdot}375-0{\cdot}75 = 0$. Hence we obtain the graduation of the x scale.

It is seen, therefore, that the presence of the constant $-0{\cdot}75$ entails a shifting of the zero of the x scale 0·75 of its own unit above the level of the other zeros. If the constant had been positive, the shifting of the x zero would have been downwards.

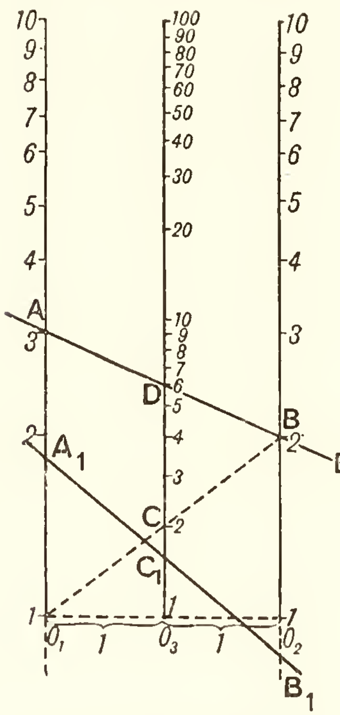

FIG. 67.—Nomogram for $x = yz$, or $x = \frac{y}{z}$.

Product and Quotient Nomogram, $x = yz$ **or** $x = \frac{y}{z}$.—Place the x scale midway between the y and z scales as in the sum and difference nomogram, except that instead of graduating the y and z scales arithmetically, **graduate them with the same unit logarithmically** (so that the points marked 1, 2, 3, . . . $-2, -3$, etc., correspond to log 1, log 2, log 3, . . . etc.), whilst the x scale is graduated logarithmically with half the unit. We then have a nomogram for $x = yz$ or $x = \frac{y}{z}$.

Thus, in the diagram (fig. 67),

$$0_3D = \tfrac{1}{2}(0_1A + 0_2B),$$

or

$$\log x = \tfrac{1}{2}(\log y + \log z) = \tfrac{1}{2}\log yz,$$

$\therefore$ the x scale being graduated with half the unit of the y and z scales we get

$$\log x = \log yz.$$

$$\therefore \quad x = yz.$$

Similarly, $O_3C_1 = \frac{1}{2}(O_1A_1 - O_2B_1)$,

or $\log x = \frac{1}{2}(\log y - \log z) = \frac{1}{2}\log\frac{y}{z}$.

Therefore, allowing for the difference in graduations of x and of y and z, we get

$$\log x = \log\frac{y}{z},$$

or

$$x = \frac{y}{z} = yz^{-1}.$$

$\therefore$ The nomogram solves the equation $x = yz^{\pm 1}$.

Examples of such equations are—

(1) Respiratory Quotient (R.Q.) $= \dfrac{\text{Vol. of } CO_2 \text{ expired}}{\text{Vol. of } O_2 \text{ inspired}}$

(2) Colour Index $= \dfrac{\text{Hæmoglobin (per cent. of normal)}}{\text{Number of red corpuscles (per cent. of normal)}}$.

(3) Coefficient of Variation (C.V.) $= \dfrac{100\sigma}{M}$ (see p. 400).

Similarly, because when $x = y^{m_1}z^{\pm m_2}$ (*i.e.* when $x = y^{m_1}z^{m_2}$, or $\left.\dfrac{y^{m_1}}{z^{m_2}}\right)$ we have $\log x = m_1 \log y \pm m_2 \log z$,

$\therefore$ if we make $\dfrac{\text{the distance between the } y \text{ and } x \text{ scales}}{\text{the distance between the } x \text{ and } z \text{ scales}} = \dfrac{m_1}{m_2}$

and graduate y, z, x logarithmically in the appropriate manner (*cf.* p. 144), we get a nomogram for $x = y^{m_1}z^{\pm m_2}$.

Similarly, in the case of $x = Cy^{m_1}z^{\pm m_2}t^{\pm m_3} \ldots w^{\pm m_n}$, where C is a numerical constant, we have

$$\log x = \log C + m_1 \log y \pm m_2 \log z \pm m_3 \log t \pm \ldots \pm m_n \log w,$$

and therefore the x scale has to be shifted log C of its own unit in a positive or negative direction, according as C is negative or positive.

The Scale Modulus.—The numbers m_1, m_2, m_3, etc., used for graduating the three scales, are called the scale moduli. **The scale modulus, therefore, is that number of units of length** (*e.g.* ins. or cms., etc.) **which represents the unit division of the scale.**

The scale moduli for the y and z scales may be arbitrarily chosen to suit the particular requirements, but the modulus of the x scale, as well as the position of that scale,

depends, as we have seen, upon the values of the other two moduli. Conversely, if we choose the length $0_1 0_2$ (*i.e.* the distance between the two external scales), the position of the point 0_3 (marking the position of the intermediate scale) and the modulus of one of the scales, then the graduations of the other two scales become fixed. Thus, if the moduli for the y and z scales are m_1 and m_2 respectively, then

$$\frac{0_1 0_3}{0_3 0_2} = \frac{m_1}{m_2}$$

and m_3 (modulus for x scale) $= \dfrac{m_1 m_2}{m_1 + m_2}$.

EXAMPLES.

(1) Construct a nomogram for the equation

$$S = 71 \cdot 84 W^{0 \cdot 425} H^{0 \cdot 725} \text{ (Du Bois).}$$

(S = surface of body in square centimetres, W = weight in kilograms, and H = height in centimetres.)

It is obvious that in an equation like this both W and H must have certain limits. Let us therefore take tho upper and lower limits of W as 70 and 2 kgrms. respectively, and the upper and lower limits of H as 180 and 50 cm. respectively. (Fig. 68.)

Taking logarithms of both sides we have

$$\log S - \log 71 \cdot 84 = 0 \cdot 425 \log W + 0 \cdot 725 \log H.$$

If now we take three parallel lines to represent the three scales (placing the S scale between the W and H scales), this equation enables us

(*a*) To find the exact position of the S scale (*i.e.* its distance from the W and H scales, respectively).

(*b*) To graduate the three scales.

(*a*) *Position of* S *Scale*.—Since the range of W is between 2 and 70 kgrms.,

$\therefore$ the whole length of the W scale will be

$$0 \cdot 425 (\log 70 - \log 2) = 0 \cdot 425 (1 \cdot 8451 - 0 \cdot 3010)$$
$$= 0 \cdot 656 \text{ of a unit of the W scale.}$$

Now, a suitable length for this range will be about 10 ins.

$\therefore$ One complete unit of the scale, *i.e.* the range of weights between 0 and 10 or between 10 and 100 kgrms. (since log 10 = 1 and log 100 = 2), will be represented by about $\dfrac{10}{0 \cdot 656} = 15 \cdot 24$ ins., *i.e.* the **modulus** of the weight scale is 15·24.

Again, the range of H is between 50 and 180 cm.,

$\therefore$ whole length of the H scale will be

$$0 \cdot 725 (\log 180 - \log 50) = 0 \cdot 725 (2 \cdot 2553 - 1 \cdot 6990)$$
$$= 0 \cdot 403 \text{ of a unit of the H scale.}$$

Taking the length suitable for this range as about 9 ins., we shall get the **modulus** for the height scale as $\frac{9}{0{\cdot}403}$ = about 22·33 ins.

$$\therefore \quad \text{Ratio between moduli} = \frac{15{\cdot}24}{22{\cdot}33} = 1 : 1{\cdot}47 \text{ (approximately),}$$

$$i.e. \quad \frac{\text{Distance between W and S scales}}{\text{Distance between S and H scales}} = \frac{1}{1{\cdot}47}.$$

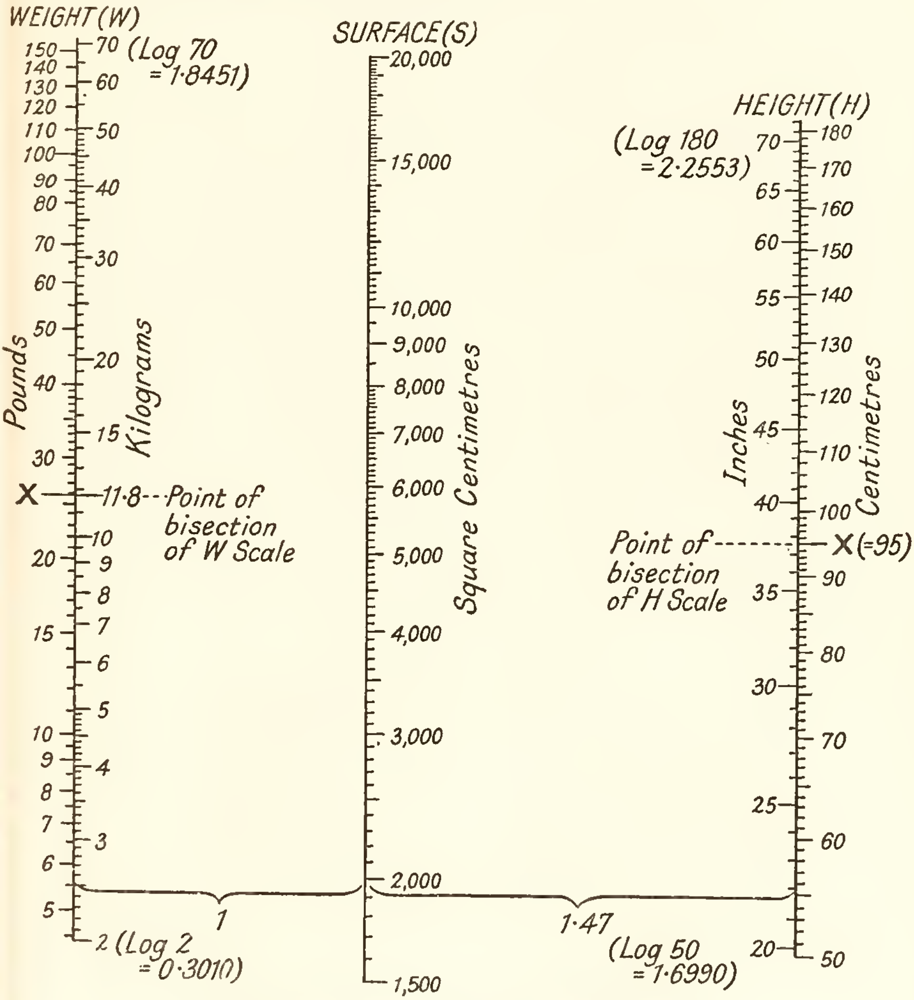

FIG. 68.—Nomogram for $S = 71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}$.

(*b*) *Graduation of the Scales.*—As the zeros of the scales are not on the chart the best way is the following:—

(1) Find the value of W corresponding to the middle point of the W scale by taking half the sum of the logarithm limits.

$$\text{Now } \tfrac{1}{2}(\log 70 + \log 2) = \tfrac{1}{2}(1{\cdot}8451 + 0{\cdot}3010)$$
$$= \tfrac{1}{2} \times 2{\cdot}1461 = 1{\cdot}0731 = \log 11{\cdot}83.$$

∴ Middle point of scale corresponds to 11·83 kgrms. (Put a small X near it for reference.)

∴ The point representing 10 kgrms. will be

$$15{\cdot}24 \times 0{\cdot}425 (\log 11{\cdot}83 - \log 10) \text{ ins.}$$

below the central point, *i.e.* 0·473 in. below.

Now calculate the position of every 5-kgrm. variation in weight above and below 10 until the top and bottom of the scale are obtained. This is done as follows:—

Number of Kilos.	5	15	20	25	30	35	40
Corresponding log Log 10 . .	0·699 1·000	1·1761 1	1·3010 1	1·3979 1	1·4771 1	1·5441 1	1·6021 1
Difference in logs.	−0·301	0·1761	0·3010	0·3979	0·4771	0·5441	0·6021

etc.

The points representing these weights will be the logarithm differences multiplied by 15·24 × 0·425, *i.e.* by 6·48, viz.:

	$6{\cdot}48 \times (-0{\cdot}301)$,	$6{\cdot}48 \times 0{\cdot}1761$,	$6{\cdot}48 \times 0{\cdot}301$,	$6{\cdot}48 \times 0{\cdot}398$,	etc.,
i.e.	$-1{\cdot}95$,	$1{\cdot}14$,	$1{\cdot}95$,	$2{\cdot}58$,	etc.,

ins. distant from the 10-kgrm. point, and so on for other points.

(2) Now graduate the H scale in the same way.

$$\text{The middle point} = \tfrac{1}{2}(\log 180 + \log 50)$$
$$= \tfrac{1}{2}(2{\cdot}2553 + 1{\cdot}6990)$$
$$= \tfrac{1}{2} \times 3{\cdot}9543 = 1{\cdot}9772$$
$$= \log 94{\cdot}9,$$
$$\text{say, } 95 \text{ cm.}$$

(Put a small X near this point for reference.)

Calculate the position of every 5-cm. variation in height above and below 95 cm. until the top and bottom of the scale are obtained. Thus:

Number of cms.	80	85	90	100	105	110
Corresponding log Log 95 . .	1·9031 1·9777	1·9294 1·9777	1·9542 1·9777	2 1·9777	2·0212 1·9777	2·0414 1·9777
Difference in logs	−0·0746	−0·0483	−0·0235	0·0223	0·0435	0·0637

Then the points representing these heights will be the logarithm differences multiplied by 22·33 × 0·725, *i.e.* by 16·19, viz.:

$16{\cdot}19 \times (-0{\cdot}0746)$, $16{\cdot}19 \times (-0{\cdot}0483)$, $16{\cdot}19 \times (-0{\cdot}0235)$, $16{\cdot}19 \times 0{\cdot}0223$,
i.e. $-1{\cdot}21$, $-0{\cdot}78$, $-0{\cdot}38$, $0{\cdot}36$,
$16{\cdot}19 \times 0{\cdot}0435$, etc.,
$0{\cdot}70$, etc.,

ins. from the 95-cm. point; and so on for the other points.

(3) The S scale is graduated by calculating a value for S from the formula

$$\begin{aligned} S &= 71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}, \text{ for values of,} \\ \text{say, } W &= 4 \text{ and } H = 60, \text{ which gives} \\ \log S &= \log 71{\cdot}84 + 0{\cdot}425 \log 4 + 0{\cdot}725 \log 60 \\ &= 1{\cdot}8563 + 0{\cdot}2558 + 1{\cdot}2892 \\ &= 3{\cdot}4013 = \log 2520. \end{aligned}$$

∴ A line joining the scales of W and H at the points 4 and 60 respectively will meet the scale S at a point which should be marked 2520, or it can without very great error be marked 2500.

The other points at intervals of 500 sq. cm. may then be calculated exactly in the same way as in the case of the other two scales, except that the differences in the logarithms must be multiplied by m_3 which $=$

$$\frac{m_1 m_2}{m_1 + m_2} = \frac{22{\cdot}33 \times 15{\cdot}24}{22{\cdot}33 + 15{\cdot}24} = 9{\cdot}0.$$

Thus the point 2000 will be

$$\begin{aligned} 9(\log 2500 - \log 2000) &= 9(3{\cdot}3979 - 3{\cdot}3010) \\ &= 0{\cdot}87 \text{ in. below the 2500 point;} \end{aligned}$$

and so on for the other points.

The graduations may also be obtained *automatically* by finding another value of S for any two arbitrarily chosen values of W and H, and measuring the length of that interval on the scale. Thus, for W = 4·7 kgrms. and H = 70 cm., calculation gives S ≑ 3000 sq. cm.* Therefore the line joining the W and H scales at the points 4·7 and 70 respectively will cut the S scale at a point corresponding to approximately 3000. The length of the S scale, therefore, between the points 2500 and 3000, which we shall call a, must correspond to the difference of the logarithms of these numbers, *i.e.*

$$3{\cdot}4771 - 3{\cdot}3979 = 0{\cdot}0792.$$

Hence the point on the S scale corresponding to 3500 will be above the 3000 point, at a distance equal to

$$\frac{(\log 3500 - \log 3000)}{0{\cdot}0792}a = \frac{(3{\cdot}5441 - 3{\cdot}4771)}{0{\cdot}0792}a = \frac{0{\cdot}067}{0{\cdot}0792}a = 0{\cdot}85a.$$

Similarly the point 4000 will be above the 3500 point at a distance equal to

$$\frac{(\log 4000 - \log 3500)}{0{\cdot}0792}a = \frac{(3{\cdot}6021 - 3{\cdot}5441)}{0{\cdot}0792}a = \frac{0{\cdot}058}{0{\cdot}0792}a = 0{\cdot}73a;$$

and so on for the other points.

The nomogram so constructed (see fig. 68, which has been reduced in the process of reproduction to about 40 per cent. of the original drawing) gives results to a considerable degree of accuracy. If the moduli had been made larger, or the reproduction less reduced, the various scales would have been longer, and the degree of accuracy greater. The W and H scales are also shown graduated on the other side with British units to avoid the necessity of conversion to metric units. (See Feldman and Umanski, *The Lancet*, 1922, vol. 1; also Boothby and Sandiford, *Boston*

* ≑ signifies approximately equal to.

Med. and Surg. Journ., 1922, vol. 185; the latter paper gives a comprehensive account of the use of nomographic charts for the calculation of the metabolic rate by the gasometer method; W. A. M. Smart, *The Lancet*, 1923, vol. 2; T. Uweda, *Acta Medicinalia*, 1929, vol. xii.)

(2) The total amount of heat (T) produced by a person in 24 hours is given by the equation T = 24SC, where S = surface area of the body and C = number of calories produced by 1 sq. metre of body surface per hour. Assuming S to be given by the Du Bois formula, viz. $S = 71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}$, construct a nomogram for the equation T = 24SC (fig. 69).

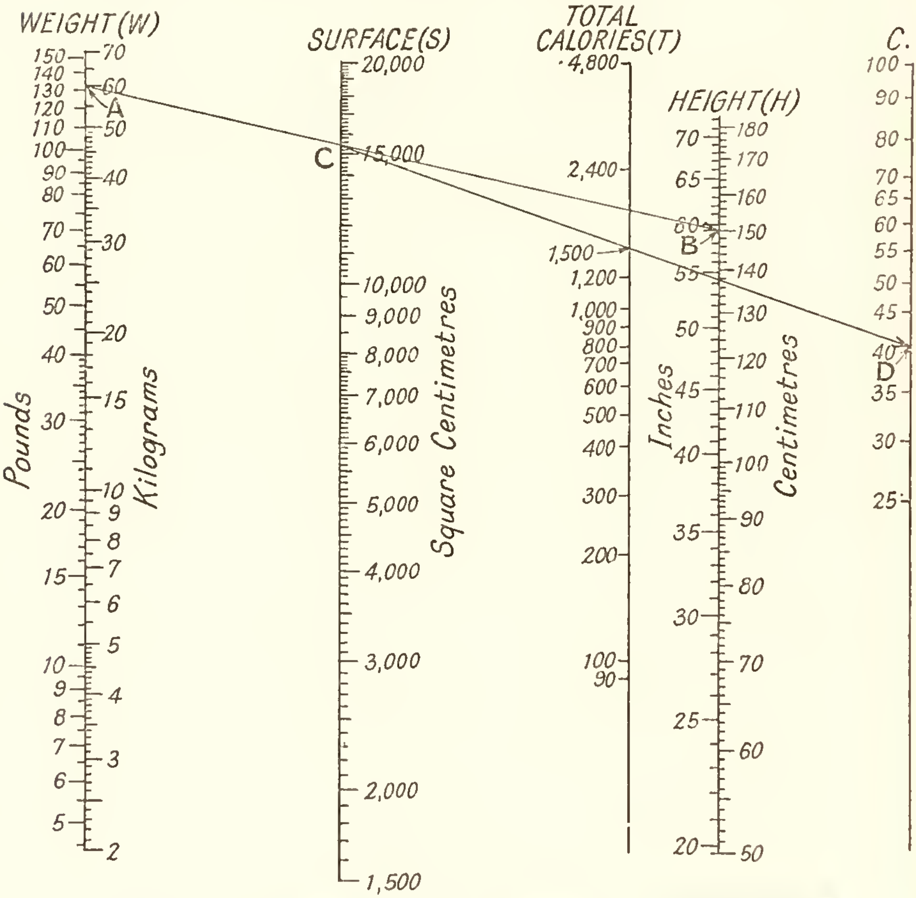

FIG. 69.—Nomogram for T = 24SC (where $S = 71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}$).

First make the nomogram for the Du Bois formula (see Example (1)). Then place the C scale at a convenient distance from the S scale, and the T scale midway between the S and C scales. The average hourly heat production per square metre of surface being about 40 calories, let us take the extreme ranges of variation of C as 25–100 calories per square metre per hour.

Graduate the C scale with the same logarithmic unit as the S scale, *i.e.* starting from any convenient level which we call 100, measure off on the C scale a distance downwards equal to that between 10,000 and 2,500 sq. cm.

on the S scale, and mark the lower point 25 (the difference between the logarithms of 10,000 and 2,500 being the same as that between the logarithms of 100 and 25). The intermediate points are marked in the same way. Thus points 30, 35, 40, etc. are placed at the same respective distances above the point 25 on the C scale, as the points 3,000, 3,500, 4,000, etc. are above the point 2,500 on the S scale.

Graduation of T scale: The join of the two highest points on the S and C scales (viz. 20,000 sq. cm. or 2 sq. metres with 100 calories) will intersect the T scale at the graduation 4,800 (because $24 \times 2 \times 100 = 4800$) which is the top point of the T scale. Similarly the point on the T scale collinear with 1500 sq. cm. or 0·15 sq. metre on the S scale and with 25 on the C scale gives the bottom point (90) of the T scale because $24 \times 0{\cdot}15 \times 25 = 90$. The intermediate points are graduated with half the S or C unit.

Method of Use: Find the total heat produced by a person whose weight = 60 kilos, whose height = 150 cm., and who produces heat at the rate of 40 calories per square metre per hour.

Join 60 (A) on the W scale with 150 (B) on the H scale. The point of intersection (C) of the joining line AB with the S scale is joined with the point 40 (D) on the C scale. The point of intersection of this joining line with the T scale gives the total number of calories produced by the person in a day. It is seen to be approximately 1500 (calculation gives 1485).

Nomograms with Intersecting and Curved Scales.—All the scales in a nomogram need not necessarily be parallel, or straight. They may intersect, and one of the scales may be curved. A consideration of these, however, is outside the scope of this book. (See "*A First Course in Nomography*," by S. Brodetsky, London, 1925.)

EXERCISES.

Construct nomograms for the following:—

(1) Respiratory Quotient, viz. R.Q. $= \dfrac{\text{Vol. of } CO_2 \text{ expired}}{\text{Vol. of } O_2 \text{ inspired}}$, the range for both CO_2 and O_2 to be 100–1000 c.c.

[Place numerator scale (Vol. of CO_2) midway between the other two scales (R.Q. and Vol. of O_2). Graduate the two external scales with the same logarithmic unit and the intermediate one automatically.]

(2) Colour Index of Blood, viz.

$$\text{C.I.} = \frac{\text{Hæmoglobin (per cent. of normal)}}{\text{No. of erythrocytes (per cent. of normal)}}$$

the range for cells being 1–10 million and that for Hb 5 to 150 per cent.

[See (1).]

(3) Du Bois formula, making the S scale inclined at an angle to the other two scales.

[Graduate the W and H scales in the ordinary way and the S scale between the other two automatically.]

CHAPTER X.

DIFFERENTIALS AND DIFFERENTIAL COEFFICIENTS.

In the case of all the functions with which we dealt in Chapter VIII. we considered the relative increments or decrements of mutually related quantities when such increments or decrements were infinitesimally small. Such infinitesimally small increments or decrements are called **differentials,** differential being the diminutive of the word difference. The differential of any quantity x is written as dx, where d is an abbreviation of the expression "the differential of." Hence, whenever one meets with the expression dx, one must not think of it as "d multiplied by x" but as "the differential of x" or as "an infinitesimally small bit of x." Similarly, dy stands for "the differential of y," or "an infinitesimally small bit of y."

dx is read as "dee-eks,"
dy is read as "dee-wy,"

and so on.

Now, although dx stands for an infinitesimally small and therefore, in itself, negligible bit of x, it does not follow that such quantities as xdx, x^2dx, etc., are negligible, but $(dx)(dx)$ or $(dx)^2$ would be negligible.

Let us take an example. Supposing $x = 1{,}000{,}000$ and $dx = 0{\cdot}000001$, then $xdx = 1$, $x^2dx = 1{,}000{,}000$, and so on, *i.e.* xdx is not a negligible quantity and x^2dx is as large as the original quantity x itself; but $(dx)^2 = (10^{-6})^2 = 10^{-12}$ is an utterly negligible quantity.

Now, if we speak of any fraction $\frac{1}{n}$ as a very small proportion of the whole, then the fraction $\frac{1}{n^2}$ would be called a fraction of the second order of smallness, or, in mathematical language, $\frac{1}{n^2}$ would be called a fraction of the second order of magnitude, and $\frac{1}{n^3}$ would be a fraction of the third order of smallness (or magnitude), and so on.

Hence, if we make $\frac{1}{n}$ small enough, *e.g.* $\frac{1}{10^{10}}$ (which, as we saw on p. 72, corresponds to about 1 inch in 200,000 miles)—or smaller still—then we are justified in neglecting such fractions of the second order of magnitude (or smallness) unless such fractions happen to occur as factors multiplied by some quantity which is itself very large.

Notation Used in Connection with Very Small Quantities.—If we want to say that a certain portion of x is so exceedingly small as to be **practically,** but **not absolutely,** negligible, we denote such a portion of x by the symbol Δx. If, however, Δx is made smaller and smaller until it becomes infinitesimally small, *i.e.* until it becomes in itself *absolutely* negligible, or zero, then we denote such a portion of x by the symbol dx, by which we mean the differential or infinitesimally small bit of x.

Hence we can say that dx is the limit of Δx when Δx is made infinitely small.

Now, if in any function $y = f(x)$, the independent variable x changes in value from x to $(x+dx)$, where dx is, as we have seen, an infinitesimally small increment (or decrement) of x, then the dependent variable y will necessarily undergo a correspondingly infinitesimal change and become $y+dy$.

The ratio between the infinitesimally small change dy in the dependent variable to the infinitesimally small change dx in the independent variable, *i.e.* the ratio $\frac{dy}{dx}$ is called the **differential coefficient** of y with respect to x. In other words, a differential coefficient represents the true rate of change at any moment (as contrasted with the average rate of change during a definite interval of time) of the dependent variable y with respect to any change of the independent variable x.

Thus if y = distance travelled by a body in time x, then $\frac{dy}{dx}$ = the true velocity of the body at any moment.

If y = length, or area, or volume, of a body that is heated to any temperature x, then $\frac{dy}{dx}$ = coefficient of expansion (linear, superficial, or cubical).

If y = concentration of any body undergoing chemical change during a period of time t, then $\frac{dy}{dx}$ = reaction velocity.

Average and Real (or True) Rate of Change.—The term **rate of change** has not exactly the same meaning in its ordinary

colloquial use as it has when used in scientific terminology. Thus, when we say, for instance, that the velocity of a train during a certain period of its journey was, say, 30 miles per hour, we mean that the distance (in miles) covered by the train during that **measurable** period (in hours), divided by the length of that period, was equal to 30.

Thus the period of observation might have been 15 minutes, or 0·25 hour, during which time the train covered a distance of $7\frac{1}{2}$ miles. We then get:

$$\text{Speed during the 15 minutes, or } \tfrac{1}{4} \text{ hour,} = \frac{7\cdot5}{0\cdot25}$$
$$= 7\cdot5 \times 4 = 30 \text{ miles per hour.}$$

But it is clear that unless the speed was uniform the velocity must have kept on changing during that measured period, so that the above calculation gives us no idea of the *actual* velocity of the train **at any moment** during that $\frac{1}{4}$ hour. What it does give us is the **average** velocity during that interval of time, and if, for instance, we had taken a different period of observation, say, 5 minutes, we might have found that during that shorter period the train covered a distance of, say, 3 miles, so that the velocity during that interval was $\dfrac{3}{1/12} = 3 \times 12 = 36$ miles per hour.

In scientific problems, as we have said, we have to investigate the instantaneous velocity, or rate of change, of some growing or varying quantity. In other words, we have to find out what is the *actual* rate of change of the dependent variable (such as distance) *at any moment*, *i.e.* during an infinitesimally small and therefore *immeasurable* change of the independent variable (such as time).

It might at first seem not only a contradiction, but an impossibility, to *measure* the amount of change during an *immeasurable* period, but it is the beauty of mathematics that it renders the apparently impossible not only possible, but easy. For example, we cannot reach the sun, yet we can measure its distance. We also saw in Chapter VI how it is possible to sum certain series although they contained an infinite number of terms. In what follows the object will be to show how the problem of finding *actual*, or *true*, or *instantaneous*, rates of change can be solved. We shall first take an example and work it out by logical reasoning from first principles, and, when that example has been thoroughly mastered and understood, the rules which we shall develop,

and the methods of their development, will become quite easy.

Suppose a body to be moving with a uniform velocity of, say, 10 ft. per second, and at a certain fixed instant it has imparted to it an acceleration of 6 ft. per second per second. We know from elementary mechanics that the distance S covered by that body during an interval of time t from the fixed instant is given by the formula

$$S = 10t + 3t^2.$$

Now, with the aid of this formula we can easily ascertain what is the **average** velocity of the body during any **definite measurable** interval of time. Thus, for the average velocity during the fourth second we have

Distance covered in 3 seconds $= 10 \times 3 + 3 \times 9 = 57$ ft.
Distance covered in 4 seconds $= 10 \times 4 + 3 \times 16 = 88$ ft.

$\therefore$ Distance covered during the fourth second $= 88 - 57 = 31$ ft.

$\therefore$ *Average velocity during the whole of the fourth second* $=$ 31 *ft. per second.*

Now let us see what is the average velocity during the first half of the fourth second.

Distance covered in 3 seconds $= 10 \times 3 + 3 \times 9 = 57$ ft.
Distance covered in 3·5 seconds $= 10 \times 3{\cdot}5 + 3 \times (3{\cdot}5)^2 = 71{\cdot}75$ ft.

$\therefore$ Distance covered during first half of the fourth second $= 71{\cdot}75 - 57 = 14{\cdot}75$ ft.

$\therefore$ *Average velocity during that half second* $= 14{\cdot}75/0{\cdot}5 =$ 29·5 *ft. per second.*

But this, again, does not represent the true velocity during the **whole** of the half a second, because the velocity is not constant, but is continually increasing. Let us therefore find what the average velocity is during the first $\frac{1}{10}$ of the fourth second. We have again,

Distance covered in 3 seconds $= 10 \times 3 + 3 \times 9 = 57$ ft.
Distance covered in 3·1 seconds $= 10 \times 3{\cdot}1 + 3 \times (3{\cdot}1)^2 = 59{\cdot}83$ ft.

$\therefore$ Distance covered during the first $\frac{1}{10}$ or 0·1 of the fourth second $= 59{\cdot}83 - 57 = 2{\cdot}83$ ft.

$\therefore$ Average velocity during that $\frac{1}{10}$ second $= \dfrac{2{\cdot}83}{0{\cdot}1} = 28{\cdot}3$ ft. per second.

But, as we said, this again only gives the *average* velocity during the $\frac{1}{10}$ second. But by taking the distance covered in 3·01 seconds, we can find the average velocity during the first $\frac{1}{100}$ of the fourth second. Thus:

Distance covered in 3·01 seconds $= 10 \times 3{\cdot}01 + 3 \times (3{\cdot}01)^2 =$ 57·2803 ft.

$\therefore$ Distance covered in first $\frac{1}{100}$ of the fourth second

$$= 57{\cdot}2803 - 57$$
$$= 0{\cdot}2803 \text{ ft.}$$

$\therefore$ Average velocity during that $\frac{1}{100}$ or 0·01 of a second

$$= \frac{0{\cdot}2803}{0{\cdot}01} = 28{\cdot}03 \text{ ft. per second.}$$

Similarly, it can be shown that:

Average velocity during the first $\frac{1}{1000}$ or 0·001 of the fourth second = 28·003 ft. per second.

Average velocity during the first $\frac{1}{10{,}000}$ or 0·0001 of the fourth second = 28·0003 ft. per second.

Average velocity during the first $\frac{1}{100{,}000}$ or 0·00001 of the fourth second = 28·00003 ft. per second.

Average velocity during the first $\frac{1}{1{,}000{,}000}$ or 0·000001 of the fourth second = 28·000003 ft. per second,

and so on, there being always as many zeros preceding the 3 in the decimal as there are in the decimal of the second considered. Hence we see that if we calculate the velocity during smaller and smaller intervals of time, we find that it approaches closer and closer to 28 ft. per second, and *in the limit*, when the interval of time is infinitesimally small, *i.e.* during the time dt, the **actual** velocity of the body, or its instantaneous velocity at that instant, *becomes* 28 *ft. per second*. In other words, the actual or true or instantaneous velocity at the end of the third second or at the beginning of the fourth second of a body moving in accordance with the law $S = 10t + 3t^2$ is 28 ft. per second.

Putting in the notation of the calculus, we say that

if $$S = 10t + 3t^2,$$

then $$\frac{dS}{dt} \text{ (at the moment when } t = 3) = 28 \text{ ft. per second.}$$

Similarly, we can show that the actual velocity at the end of the fourth or beginning of the fifth second is 34 ft. per second; and so on.

Let us now see how we can arrive at a general formula giving this **actual velocity** at any moment (t) in a simple way.

We have

$$S = 10t + 3t^2.$$

If we increase t to $(t+dt)$, S will correspondingly become $(S+dS)$.

$$\begin{array}{llll}
\text{Hence} & S + dS & = & 10(t+dt) + 3(t+dt)^2 \\
& & = & 10t + 10dt + 3t^2 + 6tdt + 3(dt)^2 \\
\text{But} & S & = & 10t + 3t^2 \\
\hline
\therefore \text{ By subtraction} & dS & = & 10dt + 6tdt + 3(dt)^2
\end{array}$$

But $(dt)^2$ being of the second order of smallness may in the limit be neglected (see p. 153).

$$\therefore \quad dS = 10dt + 6tdt = (10+6t)dt.$$

$$\therefore \quad \frac{dS}{dt} = 10 + 6t.$$

This gives the actual velocity at any moment t. Thus, if $t = 3$,

$$\frac{dS}{dt} = 10 + 6 \times 3 = 28 \text{ ft. per second.}$$

Similarly, at the end of the fourth second

$$\frac{dS}{dt} = 10 + 6 \times 4 = 34 \text{ ft. per second,}$$

and so on.

The beginner is sometimes puzzled to know why an expression like $\frac{dS}{dt}$ or $\frac{dy}{dx}$, etc., should be called a differential **coefficient.** The Germans call it a differential **quotient,** which it obviously is. But if the reader will look at the expression above, viz. $dS = (10+6t)dt$, he will see at once that $(10+6t)$, which $= \frac{dS}{dt}$, is the **coefficient** of the differential dt. Hence the name.

Differentiation.—As we shall constantly have to deal with biological phenomena in which quantities keep on growing or changing, it will be our business to find the value of $\frac{dy}{dx}$ (which, as we have seen, represents the rate of change of y at any moment as x keeps on growing) for various functions of x. The method of finding the differential coefficient of a function is called *differentiation*.

Let us take as another illustration any one of the examples of the compound interest law to which we devoted so much attention in Chapter VII. The simplest case, again, is money

allowed to grow at true compound interest. Let us take the most general equation for such a form of growth, viz.:

$Q_t = Q_0 e^{kt}$ (see p. 79),

where Q_0 = original capital,

k = constant of growth = $\frac{r}{100}$ where r = interest per cent. per annum,

t = time in which the money is allowed to grow,

e = 2·71828 . . . (see p. 77),

and Q_t = amount to which Q_0 has grown in time t.

To simplify matters let us put $Q_0 = 1$, and let us designate Q_t by the letter y. Then we have here a function in which t (the time) is the independent variable and y (the amount to which the capital has grown) is the dependent variable (*i.e.* dependent upon the value of t).

Supposing we did not know what sort of growth such a function represents, and we made it our business to find out, we would clearly set out to ascertain the rate of change of y with each infinitesimally small increase in the value of t, *i.e.* we would have to find the differential coefficient of y with respect to t, or $\frac{dy}{dt}$.

Let us proceed to do so, and see the result at which we arrive:

$$y = e^{kt},$$

$\therefore$ if t becomes increased to $(t+dt)$ (where dt is infinitesimally small) then y will in consequence become $y+dy$.

Our new equation, therefore, after an infinitesimally small interval of time dt, will be

$$y+dy = e^{k(t+dt)} = e^{kt+kdt}.$$

But $\qquad y = e^{kt},$

$\therefore$ By subtraction

$$dy = e^{kt+kdt} - e^{kt} = e^{kt} \times e^{kdt} - e^{kt}$$

(see p. 6, Law I.).

$$= e^{kt}(e^{kdt} - 1).$$

Now, by the exponential theorem we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ .\ .\ .$$

(see p. 80).

$\therefore$ By putting $x = kdt$ we get

$$e^{kdt} = 1 + \frac{kdt}{1} + \frac{k^2(dt)^2}{1\,.\,2} + \frac{k^3(dt)^3}{1\,.\,2\,.\,3} + \ .\ .\ .$$

But $(dt)^2$, being of the second order of magnitude, may in the limit be rejected, as may also be all the other terms containing $(dt)^3$, $(dt)^4$, etc.

$$\therefore \quad e^{kdt} \text{ ultimately becomes } = 1 + kdt.$$

$$\therefore \quad e^{kdt} - 1 = kdt.$$

Hence we get

$$dy \text{ [which equals } e^{kt}(e^{kdt} - 1)] = e^{kt} \,.\, kdt = ke^{kt} \,.\, dt.$$

$$\therefore \quad \frac{dy}{dt} = ke^{kt} = ky \text{ (since } y = e^{kt}).$$

In other words, the differential coefficient of this function, which expresses the rate of change of y with respect to t, tells us that in this particular function the growing quantity y grows in such a way that its increase in growth at any moment $\left(i.e.\ \frac{dy}{dt}\right)$ is proportional to its value at that moment.

We shall see later (p. 184) that this function can be differentiated in a much simpler way, but the method adopted here, though somewhat complicated, will repay careful study.

Note.—It is most important to notice that, as $\frac{dy}{dx}$ represents a momentary **rate of change,** therefore if $\frac{dy}{dx}$ is **+ve** the value of y **increases** with increases of x, and if $\frac{dy}{dx}$ is **−ve** the value of y **diminishes** with every increase of x. Thus in the case of the absorption of light by a transparent medium we saw that the amount of light passed through becomes less and less as the thickness of the medium increases, and that in such a case the form of the function is

$$Q_t = Q_0 e^{-kt}$$

or

$$y = e^{-kt}$$

whence

$$\frac{dy}{dt} = -ke^{-kt},$$

$$i.e.\ \frac{dy}{dt} \text{ is negative.}$$

Hence we have the following *rules*:—

(1) **In the case of any function** $y = f(x)$, **so long as** $\frac{dy}{dx}$ **is +ve for any value of** x, **then** y **keeps on increasing with every increase of** x **; but when we find** $\frac{dy}{dx}$ **is −ve for some value of** x, **then we know that with every increase of** x, y **keeps on decreasing.**

(2) As a corollary of (1) it follows that **if for any value of** x, $\frac{dy}{dx} = 0$, **then for that particular value of** x **the value of** y **is stationary.**

These useful rules are of fundamental importance in the study of maxima and minima of functions (see Chapter XI.).

Pictorial Representation of a Differential Coefficient.—To represent $\frac{dy}{dx}$ pictorially, and to understand its meaning in a geometrical sense, it is necessary to be perfectly clear about the two following points, viz.:

(1) The distinction between a geometrical tangent to a curve and the trigonometrical tangent of an angle. This has been explained on p. 34.

(2) The meaning of the term **slope** or **gradient.**

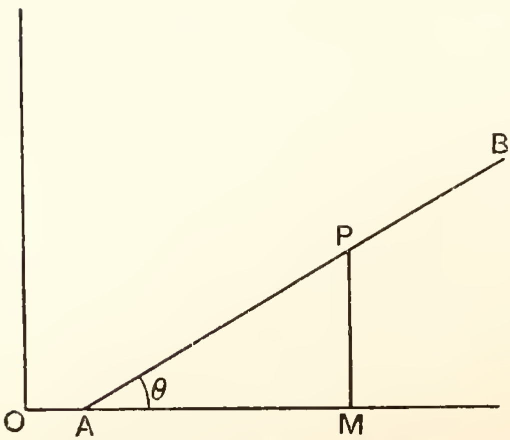

FIG. 70.—To illustrate the Meaning of Slope of a Straight Line.

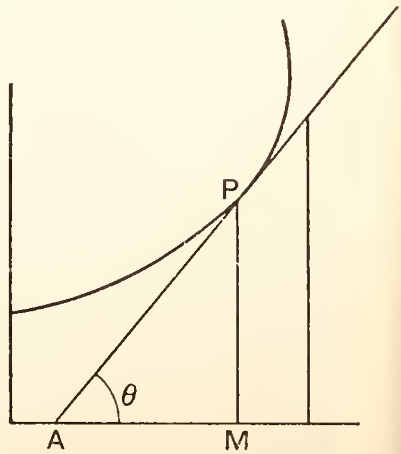

FIG. 71.—To illustrate the Meaning of the Slope of a Curve.

If we take any line AB, then its slope is expressed by the tangent (trigonometrical) of the angle which the line makes with the axis of x, *i.e.* by the ratio $\frac{\text{MP}}{\text{AM}}$ (fig. 70).

Similarly, the slope of any curve at any point P is the tangent of the angle θ which the geometrical tangent at P makes with the axis of x, *i.e.* the ratio $\frac{\text{MP}}{\text{AM}}$ (fig. 71).

Now consider any function $y = f(x)$.

Let APQ be a portion of the graph of this function (fig. 72), and let P and Q be two points very close to each other; then if the co-ordinates of P be x, y, the co-ordinates of Q will be $x + \Delta x$, $y + \Delta y$ (where Δx = a minute but finite part of x, Δy = a minute but finite part of y).

And if the chord QP makes an angle θ with the axis of x,

then the line PR, which is parallel to the axis of x, also makes an angle θ with QP.

$$\therefore \quad \tan\theta = \frac{\Delta y}{\Delta x}.$$

Now, as Q is taken nearer and nearer to P the line QP tends to become the geometrical tangent of the curve APQ at the point P or Q.

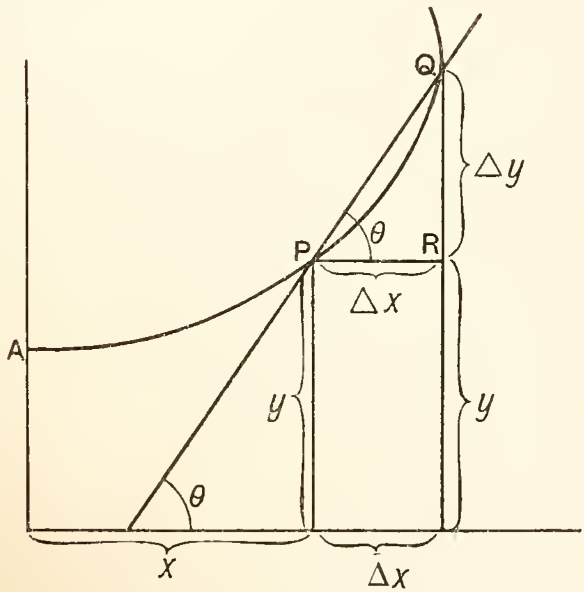

FIG. 72.—Geometrical Meaning of a Differential Coefficient.

But in the limit, when Q coincides with P, Δx becomes dx, and therefore also Δy becomes dy.

$$\therefore \quad \frac{\Delta y}{\Delta x} \text{ becomes in the limit } \frac{dy}{dx}, \textit{ i.e. } \mathrm{Lt}_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Hence $\dfrac{dy}{dx}$ is the trigonometrical tangent of the angle which the geometrical tangent of the curve at any point makes with the axis of x.

In other words, *the differential coefficient of a function* $y = f(x)$ *is the slope of the curve of that function at any definite point* (x, y).

Notation.—The differential coefficient of $f(x)$ is generally denoted by $f'(x)$.

Thus, if $$y = f(x),$$

then $$\frac{dy}{dx} = f'(x).$$

General Methods of Differentiation.

(1) **Algebraic Functions.**—*Differential coefficient of x^n.*

Let $\qquad y = x^n.$

Then $\qquad y+dy = (x+dx)^n.$

$$= x^n + \frac{nx^{n-1}}{1}dx + \frac{n(n-1)x^{n-2}}{1\,.\,2}(dx)^2 + \frac{n(n-1)(n-2)x^{n-3}}{1\,.\,2\,.\,3}(dx)^3 + \ldots$$

(by the binomial theorem, see p. 66).

But all terms to the right of $\frac{nx^{n-1}}{1}dx$ in this expansion containing quantities of the second and higher orders of magnitude may be rejected.

Hence $\qquad (x+dx)^n = x^n + \frac{nx^{n-1}dx}{1},$

i.e. $\qquad y+dy = x^n + nx^{n-1}dx.$

But $\qquad y = x^n,$

$$\therefore \quad dy = nx^{n-1}dx \text{ (by subtraction).}$$

$$\therefore \quad \frac{dy}{dx} = nx^{n-1},\ i.e.\ \frac{dx^n}{dx} = nx^{n-1}.$$

Hence we obtain the following rule: *To obtain the differential coefficient of some power of x, multiply the function by the index of the power and then reduce the power by 1.*

It is most important that the student should grasp the full significance of this statement. *What exactly is meant by the statement that the differential coefficient of $x^n = nx^{n-1}$?*

It means that if x is slightly increased in value, then the corresponding increase in value of y is nx^{n-1}.

Let us take an example.

Supposing $y = x^2$, *i.e.* y is the area of a square each of whose sides is equal to x. If x undergoes any very slight change in length, what will be the corresponding change in the area of the square?

Imagine, for instance, the square to be made of iron whose coefficient of expansion is 0·000012. If the length of the side = 1 metre, what change will the area of the square undergo if it is heated 1° C.?

Since $\quad y = x^2 = 1$ square metre,

$\therefore \quad y + \Delta y^* = (x + \Delta x)^2 = (1 + 0{\cdot}000012)^2.$

$\therefore \quad \Delta y$, *i.e.* increase in area of square,

$$= (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2$$
$$= 2 \times 0{\cdot}000012 + (0{\cdot}000012)^2.$$

As $(\Delta x)^2$, *i.e.* $(0{\cdot}000012)^2$, is a fraction of the second order of magnitude, it may be rejected in comparison with the original area x^2 (*i.e.* 1). We therefore say that as the side increases by 0·000012 metre (*i.e.* Δx) the increase in area differs from $2 \times 0{\cdot}000012$ (*i.e.* $2x\Delta x$) square metre by the practically negligible amount of $(0{\cdot}000012)^2$ square metre. If the square were heated through $\frac{1}{100}$° C. the increase (Δx) in length of each side would be 0·00000012 metre, and the increase Δy in the area of the square would differ from $2x\Delta x$, viz. $2 \times 0{\cdot}00000012$ square metre, by the still more negligible amount of $(0{\cdot}00000012)^2$ square metre. Indeed as the increase, Δx, in the side becomes less, the corresponding increase in area, Δy, becomes more nearly equal to $2x\Delta x$, and when the increase in the length of the side is infinitesimally small, *i.e.* when Δx becomes dx, the increase in area, dy, becomes exactly equal to $2xdx$, and the rate of increase in area per unit increase in length, viz. $\frac{dy}{dx}$, becomes exactly equal to $2x$ (= 2 in this instance).

From the formula $\frac{dx^n}{dx} = nx^{n-1}$ we obtain the following differential coefficients:—

$$\frac{dx^1}{dx} = 1 \,.\, x^0 = 1,$$

$$\frac{dx^2}{dx} = 2x,$$

$$\frac{dx^3}{dx} = 3x^2,$$

$$\frac{dx^4}{dx} = 4x^3,$$

and so on.

This formula holds good whether n be positive or negative, integral or fractional.

* Whilst dx or dy represents an infinitesimally small change of x or of y, Δx and Δy are used to denote very minute but not infinitesimally small changes of the variables.

Thus, let $n = -m$,

then $y = x^{-m}$.

$$\therefore \quad y + dy = (x+dx)^{-m}$$

$$= x^{-m} - mx^{-m-1}dx + \frac{m(m+1)}{1 \cdot 2}x^{-m-2}(dx)^2 - \ . \ . \ .$$

$\therefore \quad dy = -mx^{-m-1}dx \pm$ some higher powers of dx which may be rejected.

$$\therefore \quad \frac{dy}{dx} = -mx^{-m-1}.$$

Thus, if $y = x^{-1}\left(i.e.\ \frac{1}{x}\right)$,

then $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$.

If $y = x^{-2}$,

then $\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$.

If $y = x^{-3}$,

then $\frac{dy}{dx} = -3x^{-4} = -\frac{3}{x^4}$,

and so on.

Let n be fractional $= \frac{1}{m}$,

then $y = x^{\frac{1}{m}}$.

$$\therefore \quad y + dy = (x+dx)^{\frac{1}{m}}$$

$= x^{\frac{1}{m}} + \frac{1}{m}x^{\frac{1}{m}-1}dx +$ some higher powers of dx which may be rejected.

$$\therefore \quad dy = \frac{1}{m}x^{\frac{1}{m}-1}dx.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{m}x^{\frac{1}{m}-1}.$$

E.g., if $y = x^{\frac{1}{2}}$,

then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

If
$$y = x^{\frac{1}{3}}$$
then
$$\frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}},$$
and so on.

Comparison of the Geometrical with the Algebraical Method of finding $\frac{dy}{dx}$.—Plot the graph of any function, say, $y = x^2$, as in

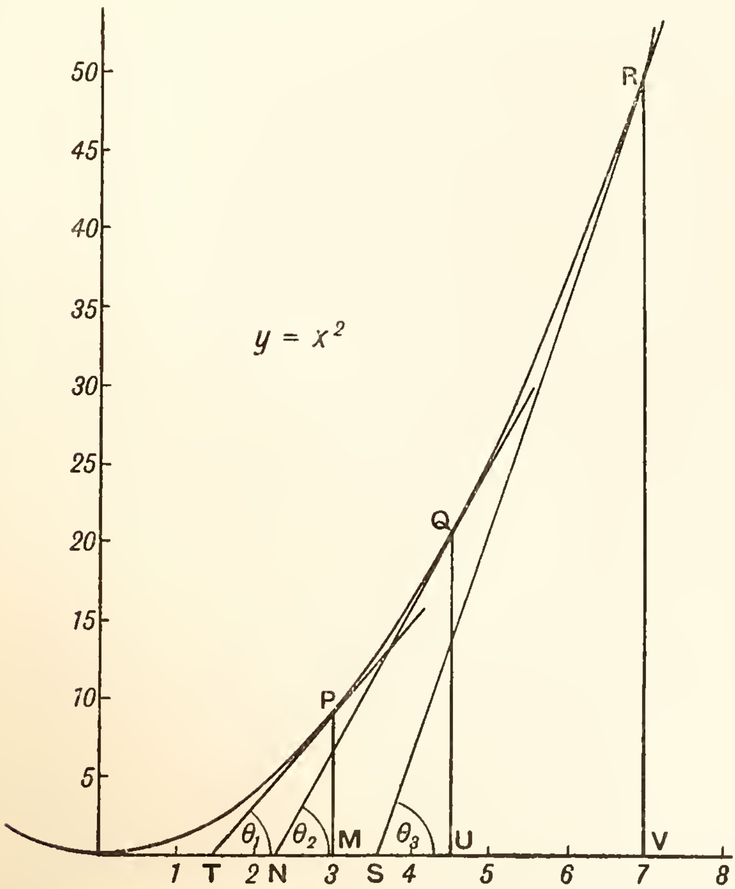

FIG. 73.—Graphical Method of showing that if $y = x^2$, then $\frac{dy}{dx} = 2x$.

fig. 73 (a parabola), and draw tangents at various points P, Q, R, whose co-ordinates are, say, (3, 9), (4·5, 20·25) and (7, 49), respectively. Measure the tangents of the angles made by these tangents with the x axis, *i.e.* the ratios $\frac{PM}{MT}$, $\frac{QU}{NU}$ and $\frac{RV}{SV}$. It

will be seen from the figure that these ratios are 6, 9 and 14 respectively $\left(i.e.\ \frac{\text{PM}}{\text{MT}}=\frac{9}{1\cdot5}=6,\ \frac{\text{QU}}{\text{NU}}=\frac{20\cdot25}{2\cdot25}=9, \text{ and } \frac{\text{RV}}{\text{SV}}=\frac{49}{3\cdot5}=14\right)$.

Now we know that $$\frac{dx^2}{dx} = 2x.$$

$\therefore$ When $x = 3,\quad \frac{dx^2}{dx}$ should $= 6$;

when $x = 4\cdot5,\quad \frac{dx^2}{dx}$ should $= 9$;

when $x = 7,\quad \frac{dx^2}{dx}$ should $= 14$.

Hence we see that $\frac{dy}{dx}$ at any point in the curve is truly represented by the tangent of the angle made by the geometrical tangent at that point. If we plot the values of $\frac{dy}{dx}$ or y' thus found geometrically against the corresponding values of x, *i.e.* if we plot the points (3, 6), (4·5, 9), (7, 14), etc., we shall get a graph which will represent the slope or first derivative of the original or primitive curve. The graph in question will obviously be the straight line $y' = 2x$.

EXAMPLES.

(1) Find the slope of the curve $y = \frac{1}{2}x^2$ at the points (2, 2), and (3, $4\frac{1}{2}$).

From the graph (fig. 74) it is seen that the slope or the tangent at the point for which $x = 2$ is $\frac{\text{AB}}{\text{DA}} = \frac{2}{1} = 2$.

By the algebraical method we find $\frac{dy}{dx} = \frac{1}{2} \cdot 2x = x$, so that when $x = 2$, $\tan\theta_1$ becomes $= 2$.

Similarly it is seen that at the point where $x = 3$ the slope is

$$\tan\theta_2 = \frac{\text{MP}}{\text{TM}} = \frac{4\cdot5}{1\cdot5} = 3,$$

and so on.

By plotting the various values of x and y', *i.e.* the points (2, 2), (3, 3), etc., we shall get the first derivative graph, which is obviously the straight line $y' = x$.

(2) Find graphically and by calculation the angles at which the curve $3y = x^2+x-6$ cuts each of the axes.

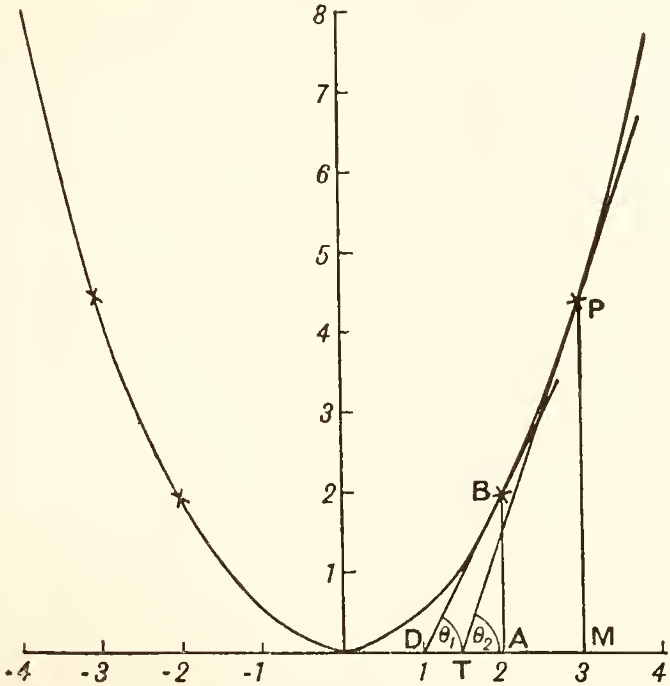

FIG. 74.—Graphical Method of Showing that if $y = \frac{x^2}{2}$, then $\frac{dy}{dx} = x$.

Graphically.—The following plotting table givos tho graph of fig. 75:—

x . . .	-4	-3	-2	-1	0	1	2	3
$y\left(=\frac{x^2+x-6}{3}\right)$.	2	0	$-1{\cdot}33$	-2	-2	$-1{\cdot}33$	0	2

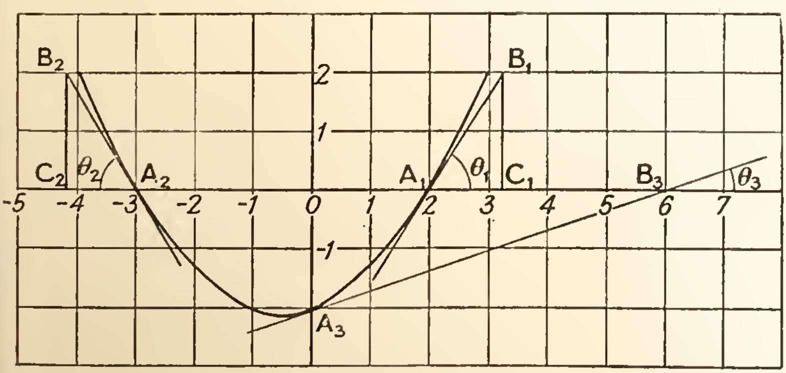

FIG. 75.

The graph cuts the abscissa axis at the points A_1 and A_2, *i.e.* where $x = 2$ and $x = -3$, respectively. It also cuts the y axis at A_3, where $y = -2$.

$$\text{The angle } \theta_1 = \tan^{-1}\frac{C_1B_1}{A_1C_1}$$

$$= \tan^{-1}\frac{2}{1\cdot 2} \text{ or } \tan^{-1} 1\cdot 666 \ldots = 59°2'.$$

$$\text{The angle } \theta_2 = \tan^{-1}\frac{C_2B_2}{A_2C_2} = \tan^{-1}\left(-\frac{2}{1\cdot 2}\right) = \tan^{-1}(-1\cdot 666 \ldots)$$

$$= -59°2'.$$

$$\text{Angle } \theta_3 = \tan^{-1}\frac{A_3O}{OB_3} = \tan^{-1}\frac{2}{6} = \tan^{-1} 0\cdot 333 \ldots = 18°26'.$$

By Calculation. $\dfrac{dy}{dx} = \dfrac{2x+1}{3}$;

$$\therefore \text{ when } x = 2, \quad \frac{dy}{dx} = \frac{5}{3} = 1\cdot 666 \ldots$$

$$,, \quad x = -3, \quad \frac{dy}{dx} = -\frac{5}{3} = -1\cdot 666 \ldots$$

$$,, \quad x = 0, \quad \frac{dy}{dx} = \frac{1}{3} = 0\cdot 333 \ldots$$

(3) A circular metal disc of radius 4 cm. is being heated so that the radius is increasing at the rate 0·01 cm. per hour; find the rate at which the surface area is expanding.

$$\left[\frac{dS}{dr} = 2\pi r, \quad \therefore \quad dS = 2\pi r dr, \quad \therefore \quad \frac{dS}{dt} = 2\pi r \cdot \frac{dr}{dt} = 8\pi \times 0\cdot 01\right.$$

$$\left.\left(\text{since } \frac{dr}{dt} = 0\cdot 01\right) = 0\cdot 251 \text{ sq. cm. per hour.}\right]$$

(4) The following gives observations of Centigrade temperature of some water in a vessel at intervals of 5 minutes: 80°, 67°, 56°, 47°, 40°, 35°, 31°. Draw the appropriate graph, and estimate as nearly as possible the rate at which the temperature was falling at the moments when the temperature was 60°, 50°, and 40°, respectively. Taking the room temperature as 20° C., state whether the rate of cooling of the water was proportional to the excess of temperature above that of the room.

[The gradients at the given points on the graph are 2·2, 1·8, and 1·2, respectively, which are the respective rates of cooling per minute. Hence rate of cooling was approximately proportional to excess of temperature above that of room, viz. to 40°, 30° and 20° C. respectively.]

(5) A man 6 ft. high is standing at night 4 ft. away from a mast up which is being raised at the rate of 3 ft. per second a lighted lantern. At what rate does the shadow of his head approach the man when it is distant 4 ft. from him?

In fig. 76 E and F are two positions of the lantern on the mast at an interval of Δt, so that if OE is represented by y, OF will be $y + \Delta y$.

Similarly, the two corresponding positions of the head's shadow, B and C, will be represented by x and $x - \Delta x$.

Now from the properties of similar triangles we have

$$\frac{AB}{OB} = \frac{AD}{OE}, \quad i.e. \quad \frac{x-4}{x} = \frac{6}{y} \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{AC}{OC} = \frac{AD}{OF}, \quad i.e. \quad \frac{(x-4)-\Delta x}{x-\Delta x} = \frac{6}{y+\Delta y} \quad . \quad . \quad (2)$$

From (1) we get $\qquad xy - 4y = 6x \quad . \quad . \quad . \quad . \quad (1a)$

From (2) we get

$$xy - 4y - y\Delta x + x\Delta y - 4\Delta y - \Delta y\Delta x = 6x - 6\Delta x \quad . \quad . \quad (2a)$$

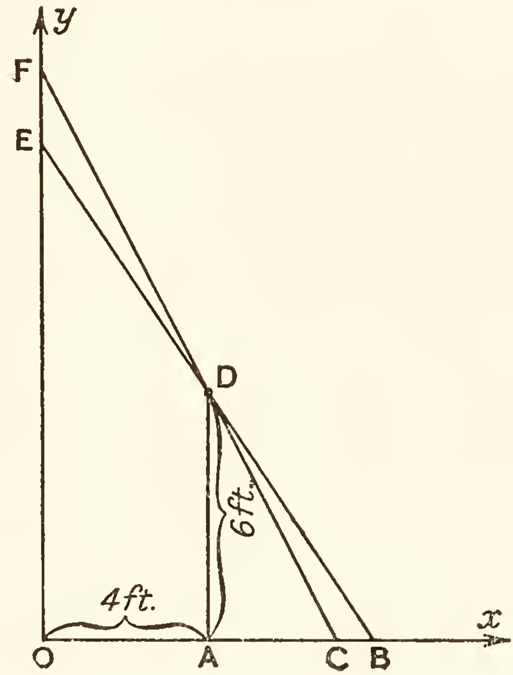

FIG. 76.

From (1a) and (2a) we get

$$-y\Delta x + x\Delta y - 4\Delta y - \Delta y\Delta x = -6\Delta x,$$

or, since $\Delta y\Delta x$ is a quantity of the 2nd order of smallness, which may be ignored,

$$y\Delta x - x\Delta y + 4\Delta y = 6\Delta x \quad . \quad . \quad . \quad . \quad (3)$$

Let us make AB $= 4$, in accordance with the data in the problem.

Then OB $= x = 8$; $\therefore$ from (1) $\frac{4}{8} = \frac{6}{y}$, whence $y =$ OE $= 12$ ft.

$\therefore$ From (3), $12\Delta x - 8\Delta y + 4\Delta y = 6\Delta x$, or $6\Delta x = 4\Delta y$.

$$\therefore \quad \Delta x = \tfrac{2}{3}\Delta y.$$

Now rate of movement of lantern $= \frac{\Delta y}{\Delta t}$, which in the limit $= \frac{dy}{dt}$,

and ,, ,, ,, shadow $= -\frac{\Delta x}{\Delta t}$,, ,, $= -\frac{dx}{dt}$.

But $\frac{\Delta y}{\Delta t} = 3$ ft. per second.

$$\therefore \quad \frac{\Delta x}{\Delta t} \quad \text{or} \quad \frac{dx}{dt} = 2 \text{ ft. per second,}$$

i.e. at the moment when the head's shadow is 4 ft. away from the man, it is moving towards him at the rate of 2 ft. per second.

EXERCISES.

(1) Draw the graph $y = \frac{1}{2}x(x+1)$ and find the values of $\frac{dy}{dx}$ at the points (4, 10), (8, 36).

[*Answer*, 4·5, 8·5.]

(2) At what point on the graph $y = 1{\cdot}8x^2$ is the tangent inclined at an angle of 45° with the x axis?

$$\left[\textit{Answer}, \ \frac{dy}{dx} = 3{\cdot}6x = \tan\theta = 1, \quad \therefore \quad x = 0{\cdot}28.\right]$$

Differentiation of a Constant.—Since a constant means a quantity which does not vary, therefore if $y = c$, $dy = 0$.

$$\therefore \quad \frac{dy}{dx} = 0,$$

i.e. **the differential coefficient of a constant = 0.**

Hence, if $y = x^n + c$, then $\frac{dy}{dx} = nx^{n-1}$.

In other words, the differential coefficient of x^n is the same as that of $x^n + c$.

Differentiation of the Product of a Constant and a Function.—

Let $\quad y = ax^n$,

$$\therefore \quad y + dy = a(x+dx)^n$$

$$= a\left(x^n + nx^{n-1}dx + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(dx)^2 + \ . \ . \ .\right).$$

$= a(x^n + nx^{n-1}dx)$, the terms to the right, being of the second and higher orders of magnitude, being rejected.

$\therefore$ By subtraction $dy = anx^{n-1}dx$,

$$\therefore \quad \frac{dy}{dx} = a \cdot nx^{n-1}.$$

But $$nx^{n-1} = \frac{dx^n}{dx}.$$

$$\therefore \quad \frac{d(ax^n)}{dx} = a \cdot \frac{dx^n}{dx};$$

i.e. **the differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.**

Hence also, if $y = ax^n + b$, then $\frac{dy}{dx} = a \cdot nx^{n-1}$.

That the differential coefficient of a constant $= 0$ can also be proved from the formula $\frac{d(ax^n)}{dx} = anx^{n-1}$. For, since $x^0 = 1$,

$$\therefore \quad c = cx^0.$$

$$\therefore \quad \frac{dc}{dx} = \frac{d(cx^0)}{dx} = 0 \cdot cx^{-1} = \frac{0}{x} = 0.$$

EXAMPLES.

(1) It has been found that the relation between the percentage (x) of casein dissolved in a given solution of alkali, and the time (t) of stirring, is expressed by the equation $x = \mathrm{K}t^m$. Find the rate at which the solution occurs (K = constant).

Differentiating we get $\qquad \frac{dx}{dt} = \mathrm{K}mt^{m-1}$

$\therefore$ velocity of solution $= \mathrm{K}mt^{m-1}$.

(2) At a given instant the radius of a soap bubble is increasing at the rate of 2 ins. per minute. What is the rate of the increase of volume when the radius is 3 ins.?

Volume of bubble, V, $= \frac{4}{3}\pi r^3$ (see p. 55).

$$\therefore \quad \frac{d\mathrm{V}}{dr} = \tfrac{4}{3}\pi \cdot 3r^2 = 4\pi r^2.$$

$$\therefore \text{ when } \quad r = 3 \text{ ins.}, \frac{d\mathrm{V}}{dr} = 4\pi \cdot 9 = 36\pi.$$

$$\therefore \quad d\mathrm{V} = 36\pi dr.$$

$$\therefore \text{ when } dr = 2 \text{ ins. per min. } d\mathrm{V} = 72\pi.$$
$$= 72 \times 3{\cdot}1416.$$
$$= 226{\cdot}2 \text{ cub. ins. per minute.}$$

(3) At what rate does the amount of light passing through the iris diaphragm of a microscope change with increase in the radius of the aperture?

Let $\quad$ radius of aperture $= r$ (fig. 77).

$\therefore$ area of aperture A $= \pi r^2$.

$$\therefore \quad \frac{d\mathrm{A}}{dr} = 2\pi r.$$

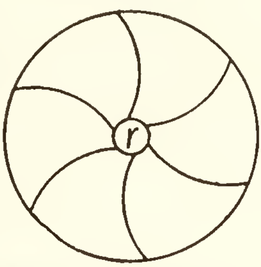

FIG. 77.—Diagram of Iris Diaphragm.

But the amount of light passing through the aperture is proportional to the area of the aperture.

$\therefore$ the rate at which the amount of light passing through the aperture changes on opening the diaphragm $= 2\pi r$.

Thus when $\qquad r = 5$ mm.,

$$\frac{dA}{dr} = 10\pi \text{ sq. mm.} = 31{\cdot}4 \text{ sq. mm.}$$

When $\qquad r = 6$ mm.

$$\frac{dA}{dr} = 12\pi \text{ sq. mm.} = 37{\cdot}7 \text{ sq. mm.},$$

and so on.

The Differentiation of an Algebraic Sum.—Let $y = x^2+x+1$. Suppose x to increase to $(x+dx)$, then y becomes $(y+dy)$.

$$\begin{aligned} \therefore \quad y+dy &= (x+dx)^2+(x+dx)+1 \\ &= x^2+2xdx+(dx)^2+(x+dx)+1. \end{aligned}$$

But
$$\begin{aligned} y &= x^2+x+1, \\ \therefore \quad dy &= 2xdx+dx+(dx)^2 \\ &= 2xdx+dx, \\ \therefore \quad \frac{dy}{dx} &= 2x+1. \end{aligned}$$

But $\quad 2x$ is the differential coefficient of x^2,
$\qquad 1 \quad$,, $\quad$,, $\quad$,, $\quad x$,
and $\quad 0 \quad$,, $\quad$,, $\quad$,, $\quad 1$.

∴ the differential coefficient of an algebraic sum is the algebraic sum of the differential coefficients of each of the terms.

Note.—It is most important for the student to realise and remember the fact that all functions of x which only differ in respect of a constant term have the same differential coefficient (see p. 238).

Thus
$$\begin{aligned} y &= 4x^5+3x^4+2x^3+7x^2+3x \\ y &= 4x^5+3x^4+2x^3+7x^2+3x+1 \\ &\ldots\ldots\ldots\ldots \\ y &= 4x^5+3x^4+2x^3+7x^2+3x+n \end{aligned}$$

when differentiated give the same result,

viz.
$$\frac{dy}{dx} = 20x^4+12x^3+6x^2+14x+3.$$

The importance of this fact will be appreciated when we have to deal with integration, *i.e.* when we are confronted with the problem of ascertaining from the differential coefficient what the original function was which gave rise to the particular differential coefficient.

Thus $\quad$ if $\quad y = x^n \quad$ or $\quad x^n+C$ (where C = any constant),

then $\quad \dfrac{dy}{dx} = nx^{n-1}$.

∴ if we come across the expression $\dfrac{dy}{dx} = nx^{n-1}$ we know that the original expression was either simply x^n or x^n+C, and we therefore always say, provisionally, that the function whose differential coefficient is nx^{n-1} is x^n+C, and we set out to find, from the other data in the problem, the value of C. It may turn out to be zero, and then we know that the original function was $y = x^n$. (See Chapter on Integral Calculus, p. 237 *et seq.*)

The Differentiation of a Product of Two Functions.—Suppose we have to differentiate the product $y = uv$, where u and v are functions of x.

Then
$$y + dy = (u + du)(v + dv)$$
$$= uv + vdu + udv + du \,.\, dv.$$
$$\therefore \quad dy = vdu + udv + du \,.\, dv.$$

But $du \,.\, dv$, being a quantity of the second order of magnitude, may be discarded.

$$\therefore \quad dy = vdu + udv$$
$$\therefore \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Hence **the differential coefficient of a product is equal to the sum of the products of each function by the differential coefficient of the other.**

E.g. let
$$y = (ax^2 + b)(cx^3 + d).$$

Now the differential coefficient of the first factor $= 2ax$ and the differential coefficient of the second factor $= 3cx^2$.

$$\therefore \quad \frac{dy}{dx} = 2ax(cx^3 + d) + 3cx^2(ax^2 + b)$$
$$= 2acx^4 + 2axd + 3acx^4 + 3cbx^2$$
$$= 5acx^4 + 3bcx^2 + 2adx.$$

(See note on p. 174.)

The Differentiation of a Product of More than Two Functions.—Let $y = uvw$, where u, v and w are functions of x. Then by treating u as one factor and vw as another factor

we get
$$\frac{dy}{dx} = u\frac{d(vw)}{dx} + vw\frac{du}{dx}.$$

But
$$\frac{d(vw)}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx}.$$
$$\therefore \quad u\frac{d(vw)}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx}.$$
$$\therefore \quad \frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.$$

Similarly, **in the case of a product of n functions, the differential coefficient of the product is equal to the sum of n products of the differential coefficient of each of the functions multiplied by the remaining $(n-1)$ functions.**

Thus if $n = 5$, we have

$$y = uvwst \text{ (say)};$$

then
$$\frac{dy}{dx} = uvws\frac{dt}{dx} + uvwt\frac{ds}{dx} + uvst\frac{dw}{dx} + uwst\frac{dv}{dx} + vwst\frac{du}{dx}.$$

Note.—The differential coefficient of a product may also in suitable cases be found by multiplying out the product and treating the result as an algebraic sum. Thus to take the same example again:

$$\begin{aligned} y &= (ax^2+b)(cx^3+d) \\ &= acx^5+bcx^3+adx^2+bd. \end{aligned}$$

$$\therefore \quad \frac{dy}{dx} = 5acx^4+3bcx^2+2adx,$$

the same result as before.

Differentiation of a Fraction.—If the fraction is such that its numerator and denominator contain a common factor, then one must first eliminate that common factor.

Thus, if

$$\begin{aligned} y &= \frac{x^2+3x+2}{x+2} \\ &= \frac{(x+2)(x+1)}{x+2} \\ &= x+1. \end{aligned}$$

$$\therefore \quad \frac{dy}{dx} = 1.$$

But supposing the fraction is given in its simplest form; how is one to differentiate it? We proceed as follows:—

Let $y = \dfrac{u}{v}$ (where u and v are functions of x).

$$= uv^{-1}.$$

Differentiating as a product (p. 173), we get

$$\begin{aligned} \frac{dy}{dx} &= v^{-1}\frac{du}{dx} + u\frac{d(v^{-1})}{dx} \\ &= \frac{1}{v}\cdot\frac{du}{dx} + u\frac{d(v^{-1})}{dx}. \end{aligned}$$

But $$\frac{d(v^{-1})}{dv} = -\frac{1}{v^2} \quad \text{(p. 162)},$$

$$\therefore \quad d(v^{-1}) = -\frac{1}{v^2}\,.\,dv,$$

$$\therefore \quad \frac{d(v^{-1})}{dx} = -\frac{1}{v^2}\cdot\frac{dv}{dx},$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{v}\cdot\frac{du}{dx} - \frac{u}{v^2}\cdot\frac{dv}{dx}$$

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Hence, the differential coefficient of a fraction is obtained as follows:—

Multiply the denominator by the differential coefficient of the numerator and subtract from this the product of the numerator by the differential coefficient of the denominator, and divide the difference by the square of the denominator.

E.g. if $$y = \frac{ax^2+b}{cx^3+d},$$

then $$\frac{dy}{dx} = \frac{(cx^3+d)2ax - (ax^2+b)3cx^2}{(cx^3+d)^2}$$

$$= \frac{2acx^4 + 2adx - 3acx^4 - 3bcx^2}{(cx^3+d)^2}$$

$$= \frac{-acx^4 - 3bcx^2 + 2adx}{(cx^3+d)^2}.$$

As an exercise, the student can verify by this method that if

$$y = \frac{x^2+3x+2}{x+2}$$

then

$$\frac{dy}{dx} = \frac{(x+2)(2x+3) - (x^2+3x+2)}{(x+2)^2} = \frac{x^2+4x+4}{x^2+4x+4} = 1. \quad \text{(See p. 174.)}$$

If we have to differentiate a fraction with a somewhat complicated denominator, then we have two courses open to us in order to simplify the process of differentiation, viz.:

(1) If the numerator and denominator contain one or more common factors, then the fraction should, as we have already stated on p. 174, be first simplified by cancelling these common factors out.

E.g. (*a*)
$$y = \frac{x^2 - 2ax + a^2}{x^3 - 3ax^2 + 3a^2x - a^3}.$$

The numerator $= (x-a)^2$.
The denominator $= (x-a)^3$.

$\therefore$ The function $y = \dfrac{(x-a)^2}{(x-a)^3} = \dfrac{1}{x-a} = (x-a)^{-1}$.

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{(x-a)^2}.$$

(*b*)
$$y = \frac{x^2 - a^2}{x^2 + 2ax + a^2} = \frac{(x-a)(x+a)}{(x+a)^2}$$
$$= \frac{x-a}{x+a}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+a)\,.\,1 - (x-a)\,.\,1}{(x+a)^2}$$
$$= \frac{2a}{(x+a)^2}.$$

(2) If there is no common factor to cancel out, then split the fraction into its partial fractions and differentiate each separately.

Thus
$$y = \frac{3x+1}{x^2-1} = \frac{1}{x+1} + \frac{2}{x-1} \quad \text{(p. 30)}.$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{(x+1)^2} - \frac{2}{(x-1)^2}$$
$$= -\frac{(3x^2 + 2x + 3)}{(x^2-1)^2}.$$

This example is given merely to illustrate the method, and not as an illustration of the type of case where splitting up into partial fractions is helpful, because in this particular instance differentiation by the ordinary rule can be done more expeditiously than by the partial fraction method. Thus

$$\frac{dy}{dx} = \frac{(x^2-1)\,.\,3 - (3x+1)\,.\,2x}{(x^2-1)^2}$$
$$= -\frac{3x^2 + 2x + 3}{(x^2-1)^2}.$$

But supposing we had a function like the following:—

$$y = \frac{(3x^2 - 2x + 1)}{(x+1)^2(x-2)}.$$

Differentiation by the ordinary method would entail a considerable amount of tedious labour, but doing it by the method of partial fractions is a comparatively simple matter. The expression when split up into its component fractions becomes

$$y = \frac{2}{(x+1)} - \frac{2}{(x+1)^2} + \frac{1}{(x-2)}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2}{(x+1)^2} + \frac{4}{(x+1)^3} - \frac{1}{(x-2)^2}.$$

Differentiation of a "Function of a Function."—In the function $y = (x^2+a^2)^{\frac{2}{3}}$, the expression (x^2+a^2) is a function of x, and the expression $(x^2+a^2)^{\frac{2}{3}}$ is a function of (x^2+a^2); therefore $(x^2+a^2)^{\frac{2}{3}}$ is a "function of a function" of x. To differentiate such a function we proceed as follows:—

$$y = (x^2+a^2)^{\frac{2}{3}}. \qquad \therefore \quad y^{\frac{3}{2}} = x^2+a^2 = u \text{ (say)}.$$

$$\therefore \quad \frac{du}{dy} \text{ or } \frac{d(x^2+a^2)}{dy} = \tfrac{3}{2}y^{\frac{1}{2}} \left(= \frac{dy^{\frac{3}{2}}}{dy}\right).$$

$$\therefore \quad d(x^2+a^2) = \tfrac{3}{2}y^{\frac{1}{2}}dy.$$

$$\therefore \quad \frac{d(x^2+a^2)}{dx} = \tfrac{3}{2}y^{\frac{1}{2}}\frac{dy}{dx}.$$

But $$\frac{d(x^2+a^2)}{dx} = 2x.$$

$$\therefore \quad \tfrac{3}{2}y^{\frac{1}{2}}\frac{dy}{dx} = 2x$$

(*i.e.* the differential coefficient of $y^{\frac{3}{2}}$ with respect to y (viz. $\frac{3}{2}y^{\frac{1}{2}}$), multiplied by the differential coefficient of y with respect to x, is equal to the differential coefficient of $y^{\frac{3}{2}}$ with respect to x).

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{\frac{3}{2}y^{\frac{1}{2}}} = \frac{4x}{3(x^2+a^2)^{\frac{1}{3}}}.$$

See also Example (9), p. 189.

In actual practice, the work is abbreviated thus:

$$y = (x^2+a^2)^{\frac{2}{3}}.$$

$$\therefore \quad y^{\frac{3}{2}} = x^2+a^2.$$

$$\therefore \quad \tfrac{3}{2}y^{\frac{1}{2}}\frac{dy}{dx} = 2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{\frac{3}{2}y^{\frac{1}{2}}} = \frac{4x}{3(x^2+a^2)^{\frac{1}{3}}}.$$

Such expressions can be dealt with somewhat differently as follows:—

Put $\qquad x^2+a^2 = u.$

Then $\qquad y = u^{\frac{2}{3}}.$

$$\therefore \quad \frac{dy}{du} = \tfrac{2}{3}u^{-\frac{1}{3}}.$$

But $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

and $$\frac{du}{dx} = 2x.$$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \tfrac{2}{3}u^{-\frac{1}{3}} \,.\, 2x \\ &= \tfrac{2}{3}(x^2+a^2)^{-\frac{1}{3}} \,.\, 2x \\ &= \frac{4x}{3(x^2+a^2)^{\frac{1}{3}}}. \end{aligned}$$

EXAMPLES.

Differentiate the following expressions:—

(1) $$y = 4x^{\frac{7}{4}}.$$
$$\frac{dy}{dx} = 4 \,.\, \tfrac{7}{4} \,.\, x^{\frac{7}{4}-1} = 7x^{\frac{3}{4}}.$$

(2) $$y = 3x^2+2x+1.$$
$$\frac{dy}{dx} = 6x+2.$$

(3) $$y = (x+a)(x+b).$$
$$\begin{aligned} \frac{dy}{dx} &= (x+a) \,.\, 1 + (x+b) \,.\, 1 \\ &= 2x+a+b. \end{aligned}$$

(4) $$y = \frac{x+a}{x+b}. \qquad \frac{dy}{dx} = \frac{(x+b) \,.\, 1 - (x+a) \,.\, 1}{(x+b)^2}$$
$$= \frac{b-a}{(x+b)^2}.$$

(5) $$y = \frac{x^m}{(x+1)^m}.$$
$$\begin{aligned} \frac{dy}{dx} &= \frac{(x+1)^m \,.\, mx^{m-1} - x^m \,.\, m(x+1)^{m-1}}{(x+1)^{2m}} \\ &= \frac{mx^{m-1}(x+1)^{m-1} \,.\, [(x+1)-x]}{(x+1)^{2m}} \\ &= \frac{mx^{m-1}}{(x+1)^{m+1}}. \end{aligned}$$

(6)
$$y = \sqrt{1+x^2}.$$
$$\therefore \quad y^2 = 1+x^2.$$
$$\therefore \quad 2y\frac{dy}{dx} = 2x.$$
$$\therefore \quad \frac{dy}{dx} = \frac{x}{y} = \frac{x}{\sqrt{1+x^2}}.$$

(7)
$$y = \sqrt{x+\sqrt{1+x^2}}.$$
$$\therefore \quad y^2 = x+\sqrt{1+x^2}.$$
$$\therefore \quad 2y\frac{dy}{dx} = 1+\frac{2x}{2\sqrt{1+x^2}} = \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$$
$$= \frac{y^2}{\sqrt{1+x^2}}.$$
$$\therefore \quad \frac{dy}{dx} = \frac{y}{2\sqrt{1+x^2}}$$
$$= \frac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}.$$

(8)
$$y = \sqrt{a+x+\sqrt{a+x+\sqrt{a+x+\text{etc.}}}}, \text{ to infinity.}$$
$$\therefore \quad y^2 = a+x+y.$$
$$\therefore \quad y^2 - y = x+a.$$

By completing the square on the left-hand side the expression becomes
$$y^2 - y + \tfrac{1}{4} = x+a+\tfrac{1}{4}$$
or
$$(y-\tfrac{1}{2})^2 = \frac{4x+4a+1}{4}.$$
$$\therefore \quad y-\tfrac{1}{2} = \pm\frac{\sqrt{4x+4a+1}}{2}.$$
$$\therefore \quad y = \frac{1\pm\sqrt{4x+4a+1}}{2}.$$
$$\therefore \quad \frac{dy}{dx} = \frac{4}{4\sqrt{4x+4a+1}} = \frac{1}{\sqrt{4x+4a+1}}.$$

(9) The Schütz-Borissoff law with regard to the action of enzymes such as pepsin and rennin is expressed by the formula
$$x = k\sqrt{\mathrm{F}at},$$
where x = amount of substance transformed,
t = time of transformation,
F = concentration of enzyme,
a = initial concentration of substrate (*e.g.* albumen or milk),
and k is a constant.

Find an expression for the velocity of hydrolysis in such a case.

Since $x = k\sqrt{Fat}$,

$\therefore \quad x^2 = k^2Fat = KFat$ (where K is another constant $= k^2$).

$$\therefore \quad t = \frac{x^2}{KFa}.$$

$$\therefore \quad \frac{dt}{dx} = \frac{2x}{KFa}.$$

$$\therefore \quad \frac{dx}{dt} = \frac{KFa}{2x} \text{ (velocity of hydrolysis),}$$

i.e. the velocity of hydrolysis in a case like this is inversely proportional to the amount of substance hydrolysed. (See, further, pp. 333 and 357.)

(10) A sandglass consists of two hollow glass cones with their apices together and their axes in a straight line. Sand runs from the upper into the lower cone through the common vertex. The height of each cone is 3 cm., and the semiapical angle is 30°.

At the beginning of the flow the sand forms a cone of depth 2 cm.

If the rate of flow of the sand is 0·5 c.c. per minute, find:

(*a*) the rate of decrease of the depth of the sand in the upper cone at the moment when the depth of the sand in the upper cone is 1 cm.,

(*b*) the corresponding rate of increase of the depth of sand in the lower cone,

(*c*) the rate at which the surface of the upper cone becomes denuded of sand,

(*d*) the rate at which the surface of the lower cone becomes covered with sand.

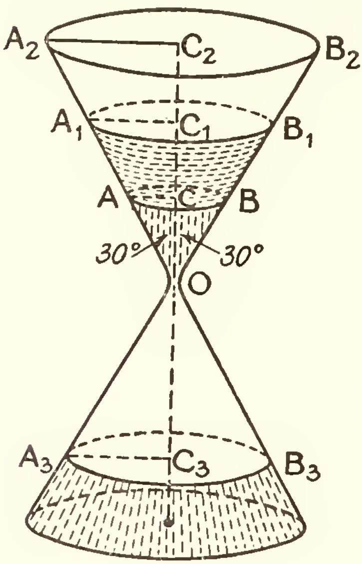

FIG. 78.

(*a*) Let OC (fig. 78) $= x =$ depth of sand (at any moment) in cm. in upper cone, and AC $= r =$ radius of section of cone at level x.

Then

$$\frac{AC}{OC}, \quad i.e. \quad \frac{r}{x}, = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore \quad r = \frac{x}{\sqrt{3}}.$$

Now volume V of a cone is given by

$$V = \tfrac{1}{3}\pi r^2 h \text{ (see p. 57).}$$

$$\therefore \quad \text{Volume of sand in OAB} = \tfrac{1}{3}\pi\left(\frac{x}{\sqrt{3}}\right)^2 x$$

$$= \frac{\pi}{9}x^3.$$

Rate of emptying of sand is $\frac{dV}{dt}$

$$= \frac{\pi}{9} \cdot \frac{dx^3}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{\pi}{3}x^2\frac{dx}{dt}.$$

But rate of emptying = 0·5 c.c. per minute,

$$\therefore \quad \frac{\pi}{3}x^2\frac{dx}{dt} = 0\cdot5, \quad \therefore \quad \frac{dx}{dt} = \frac{1\cdot5}{\pi x^2}.$$

$\therefore$ When $x = 1$, we get $\frac{dx}{dt}$ (*i.e.* rate of decrease of depth of sand) $= \frac{1\cdot5}{\pi} = 0\cdot48$ cm. per minute.

(*b*) Original volume of sand in glass when its depth was 2 cm. was

$$\tfrac{1}{3}\pi\left(\frac{2}{\sqrt{3}}\right)^2 . 2 = \tfrac{8}{9}\pi.$$

Volume of sand when depth was 1 cm. was $\frac{1}{3}\pi\left(\frac{1}{\sqrt{3}}\right)^2 . 1 = \frac{\pi}{9}$

$\therefore$ Volume of sand that ran into lower cone $= \frac{8}{9}\pi - \frac{1}{9}\pi = \frac{7\pi}{9}$.

But volume of whole lower cone $= \frac{1}{3}\pi\left(\frac{3}{\sqrt{3}}\right)^2 . 3 = 3\pi$.

$\therefore$ Volume OA_3B_3 of empty lower portion $= 3\pi - \frac{7\pi}{9} = \frac{20\pi}{9}$.

But volume $OA_3B_3 = \frac{1}{3}\pi\left(\frac{OC_3}{\sqrt{3}}\right)^2 . OC_3 = \frac{1}{9}\pi(OC_3)^3$.

$$\therefore \quad \tfrac{1}{9}\pi(OC_3)^3 = 20\frac{\pi}{9},$$

$$\therefore \quad (OC_3)^3 = 20, \quad \text{whence} \quad OC_3 = 2\cdot7144.$$

If we call the length of the axis of the empty portion of the lower cone y, we get rate of decrease of y given by $\frac{dy}{dt} = \frac{1\cdot5}{\pi y^2}$.

$$\therefore \quad \text{When } y = OC_3 = 2\cdot7144, \text{ we get } \frac{dy}{dt} = \frac{1\cdot5}{\pi(2\cdot7144)^2}$$

$$= 0\cdot065 \text{ cm. per minute.}$$

$\therefore$ The rate of increase of depth of sand in lower cone = 0·065 cm. per minute.

(*c*) The surface of a cone $= \pi r\sqrt{h^2 + r^2}$,

$$\therefore \quad \text{Surface } (s) \text{ of AOB} = \pi\frac{x}{\sqrt{3}}\sqrt{x^2 + \frac{x^2}{3}} = \frac{\pi x^2 . 2}{3} = \tfrac{2}{3}\pi x^2,$$

$$\therefore \quad \frac{ds}{dt} = \tfrac{4}{3}\pi x . \frac{dx}{dt};$$

$$\therefore \quad \text{when } x = 1, \quad \frac{ds}{dt} = \tfrac{4}{3}\pi \times 1 \times 0\cdot48 \text{ sq. cm. per minute,}$$

$$= 2\cdot01 \text{ sq. cm. per minute.}$$

(*d*) When $y = 2\cdot7144$, $\frac{ds}{dt} = \frac{4}{3}\pi \times 2\cdot7144 \times 0\cdot065 = 0\cdot74$ sq. cm. per minute.

(11) The volume of water V (in cubic feet) in a hemispherical pot of radius a feet and depth x feet is given by $V = \pi\left(ax^2 - \frac{x^3}{3}\right)$. Water is flowing into such a pot at the rate of 1 cubic foot per second. If the radius $a = 3$ feet, find the rate at which the depth of the water is increasing and calculate the rate when the water is 2 feet deep. Draw a graph of this rate of increase in depth and hence show that the depth increases very rapidly at first and then gradually more slowly.

$$V = \pi\left(ax^2 - \frac{x^3}{3}\right).$$

$$\therefore \quad \frac{dV}{dt} = \pi\left(2ax - \frac{3x^2}{3}\right)\frac{dx}{dt},$$

$$= \pi x(2a - x)\frac{dx}{dt}.$$

But $$\frac{dV}{dt} = 1, \quad \therefore \quad \frac{dx}{dt} = \frac{1}{\pi x(2a - x)};$$

$$\therefore \quad \text{when } a = 3, \quad \frac{dx}{dt} = \frac{1}{\pi x(6 - x)}.$$

Hence the rate of increase in depth when the water is 2 feet deep is

$$\frac{dx}{dt} = \frac{1}{2\pi(6-2)} = \frac{1}{8\pi} \text{ feet per second} = 0{\cdot}04 \text{ foot per second.}$$

The following plotting table gives the rate of increase of depth with increase of x:—

x . . .	0·5	1	1·5	2	2·5	3
$\pi\frac{dx}{dt}\left(= \frac{k}{6x - x^2}\right)$	0·36k	0·2k	0·15k	0·125k	0·114k	0·111k

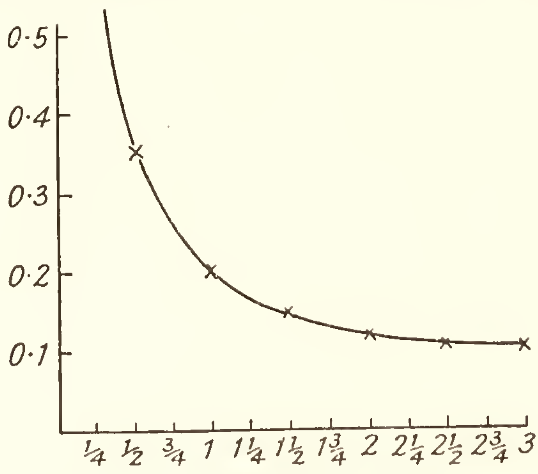

FIG. 79.

The graph (fig. 79) shows that the rate of increase in depth is most rapid at the beginning and slowest at the end.

(12) A man is walking at the rate of 4 miles an hour towards a camera in which is a lens of 6 inches focal length. When he is distant 50 feet from the camera at what rate must the ground glass be "racked" out to keep him in proper focus?

The optical formula is $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$

where f = focal length (= 6 inches or $\frac{1}{2}$ foot),
v = distance of image from lens,
u = distance of object from lens.

$$\therefore \quad \frac{1}{1/2} = \frac{1}{v} + \frac{1}{u} \quad \text{or} \quad \frac{1}{v} = \frac{1}{1/2} - \frac{1}{u} = \frac{2u-1}{u}$$

$$\therefore \quad v = \frac{u}{2u-1};$$

$$\therefore \quad \frac{dv}{dt} = \frac{(2u-1)\,.\,\frac{du}{dt} - u\,.\,\frac{d(2u-1)}{dt}}{(2u-1)^2} = -\frac{\frac{du}{dt}}{(2u-1)^2}$$

where $\frac{dv}{dt}$ is the rate at which the ground glass must be moved and $\frac{du}{dt}$ is the rate at which the man is moving = 4 miles per hour = 352 feet per minute (towards camera).

$$\therefore \quad \frac{dv}{dt} = -\frac{352}{(2u-1)^2};$$

$\therefore$ when $u = 50$ feet, $\frac{dv}{dt} = -\frac{352}{(99)^2}$ feet per minute

$= -0{\cdot}43$ inch per minute (*i.e.* in same direction as the man is moving).

(13) The velocity (V) of sound in air is $66{\cdot}3\sqrt{273+t}$ feet per second, where t is the centigrade temperature. What is the rate of rise in velocity per degree rise in temperature at 10° C.?

$$\frac{dV}{dt} = \frac{66{\cdot}3}{2\sqrt{273+t}},$$

$$\therefore \quad \text{when } t = 10^\circ \text{ C.,} \quad \frac{dV}{dt} = \frac{66{\cdot}3}{2\sqrt{283}} = 1{\cdot}97 \text{ feet per second.}$$

(14) A vessel in the shape of an inverted cone whose semivertical angle is 27° is being filled with water at the rate of 3 c.c. per second. Find the volume of the water in the cone when the depth is increasing at the rate of 2 cm. per second.

The volume of water at any level x is $\frac{1}{3}\pi r^2 x = \frac{1}{3}\pi(x \tan 27^\circ)^2 x$

$$= \tfrac{1}{3}\pi x^3 \tan^2 27^\circ$$
$$= \tfrac{1}{3}\pi x^3 (0{\cdot}5095)^2$$
$$= 0{\cdot}2718x^3.$$

Rate of filling $= \frac{dv}{dt} = \frac{dv}{dx}\,.\,\frac{dx}{dt} = 0{\cdot}8154x^2\frac{dx}{dt}.$

But $\frac{dv}{dt} = 3$ c.c. and $\frac{dx}{dt} = 2$ cm.

$\therefore \quad 3 = 1{\cdot}6308x^2$, whence $x = 1{\cdot}356$.

Therefore Volume of water $= 0{\cdot}2718x^3 = 0{\cdot}68$ c.c.

EXERCISES.

Differentiate the following:—

(1) $$y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}.$$

$$\left[\textit{Answer}, \frac{dy}{dx} = \frac{2a^2x}{\sqrt{(a^2+x^2)(a^2-x^2)^3}}.\right]$$

(2) $$y = \sqrt{x\sqrt{x\sqrt{x}}}, \text{ etc. . . . to infinity.}$$

$$\left[\textit{Answer}, \frac{dy}{dx} = 1.\right]$$

(*Cf.* Example (2), p. 22.)

(3) $$y = \cfrac{x^2}{1+\cfrac{x^2}{1+\cfrac{x^2}{1+\text{etc. } ad\ infinitum.}}}$$

$$\left[\textit{Answer}, y = \frac{x^2}{1+y}, \quad \therefore \quad y^2+y+\tfrac{1}{4} = x^2+\tfrac{1}{4}, \quad i.e. \quad (y+\tfrac{1}{2}) = \sqrt{x^2+\tfrac{1}{4}},\right.$$

$$\left.\therefore \quad \frac{dy}{dx} = \frac{x}{\sqrt{x^2+\frac{1}{4}}} = \frac{2x}{\sqrt{4x^2+1}}.\right]$$

(4) $$y = \sqrt{\frac{1-x}{1+x}}.$$

$$\left[\textit{Answer}, \frac{dy}{dx} = -\frac{1}{(1+x)\sqrt{1-x^2}}.\right]$$

Differentiation of Exponential, Logarithmic and Circular Functions.

(i) Exponential Functions.

(a) *The differential coefficient of e^x.*

The exponential series has a most important *peculiarity*, viz. that **its differential coefficient is the same as itself.** Thus,

$$\text{if} \qquad y = e^x = 1+x+\frac{x^2}{1\,.\,2}+\frac{x^3}{1\,.\,2\,.\,3}+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}+\ldots$$

$$\text{then} \quad \frac{dy}{dx} = 0+1+\frac{2x}{1\,.\,2}+\frac{3x^2}{1\,.\,2\,.\,3}+\frac{4x^3}{1\,.\,2\,.\,3\,.\,4}+\ldots$$

$$= 1+x+\frac{x^2}{1\,.\,2}+\frac{x^3}{1\,.\,2\,.\,3}+\ldots$$

$$= e^x,$$

$$i.e. \quad \frac{de^x}{dx} = e^x, \quad \text{or} \quad \frac{dy}{dx} = y.$$

This peculiarity is not shared by any other known function, and it is useful to remember it in connection with differential equations of the type $\frac{d^n y}{dx^n} = \mathrm{K}^n y$. For another peculiarity, see p. 80.

(b) *The differential coefficient of a^x.*

$$\text{Let} \qquad a = e^c.$$

$$\therefore \qquad a^x = e^{cx} = 1 + cx + \frac{(cx)^2}{1 \cdot 2} + \frac{(cx)^3}{1 \cdot 2 \cdot 3} + \ldots$$

$$\therefore \text{ if} \qquad a^x = y, \text{ we have}$$

$$\frac{dy}{dx} = 0 + c + c^2 x + \frac{c^3 x^2}{1 \cdot 2} + \ldots$$

$$= c\left\{1 + cx + \frac{(cx)^2}{1 \cdot 2} + \ldots\right\}$$

$$= ce^{cx} = ca^x.$$

$$\text{But since} \quad a = e^c, \quad \therefore \quad c = \log_e a.$$

$$\therefore \quad \frac{dy}{dx} = a^x \log_e a = \mathrm{K}y \quad (\text{where } \mathrm{K} = \log_e a).$$

In other words, **the rate of growth** $\left(\frac{dy}{dx}\right)$ **of an exponential function is proportional to itself.**

If $a > 1$, then K (which $= \log_e a$) is $+$ve, and the rate of growth of a^x is $+$ve, *i.e.* the function increases at a rate proportional to itself.

If $a < 1$, then K is $-$ve, and the rate of growth of a^x is $-$ve, *i.e.* the function diminishes at a rate proportional to itself.

Corollary.—If $y = a^x$, then $\frac{1}{y}\frac{dy}{dx}$ (*i.e.* the *proportional* rate of increase of the function) is constant.

Gradient or Slope of the Curve $y = e^x$.—If we draw the graph $y = e^x$ (see fig. 80) and draw tangents at various points on it, *e.g.* at P (0, 1), Q (1, 2·72), R (2, 7·39), making the angles θ_1, θ_2, θ_3 with the x axis, we shall find that—

$$\tan \theta_1 = \frac{1}{1} = e^0,$$

$$\tan \theta_2 = \frac{2 \cdot 72}{1} = c^1,$$

$$\tan \theta_3 = \frac{7 \cdot 39}{1} = e^2,$$

and so on, so that at any point whose abscissa is x

$$\tan \theta = e^x.$$

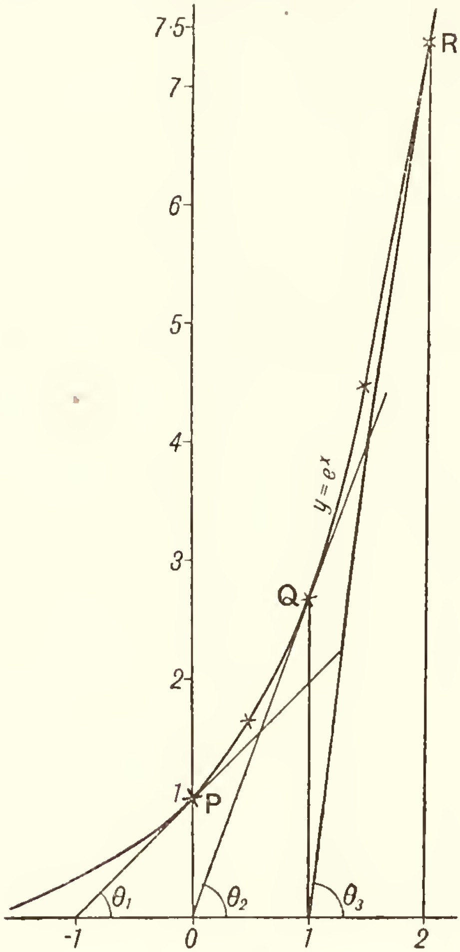

Fig. 80.—Graphical Method of showing that $\frac{de^x}{dx} = e^x$.

Hence we have a *geometrical or pictorial proof of the truth of the statement that*

$$\frac{de^x}{dx} = e^x.$$

(ii) **Logarithmic Functions.**

To find the differential coefficient of $\log_e x$.

Let $$y = \log_e x.$$

$$\therefore \quad e^y = x.$$

$$\therefore \quad \frac{dx}{dy} = e^y = x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}.$$

Corollary.—If $y = \log_e (a \pm x)$, then $\dfrac{dy}{dx} = \pm \dfrac{1}{(a \pm x)}$. For, putting $(a \pm x) = z$, we get

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{(a \pm x)} \times (\pm 1) = \pm \frac{1}{(a \pm x)}.$$

(See Example (9) for the practical application of this differential coefficient as a labour-saving device.)

EXAMPLES.

(1) If $y = \log_e (\log_e x)$, find $\dfrac{dy}{dx}$.

Let $\log_e x = u$, so that $y = \log_e u$.

$$\therefore \quad \frac{dy}{du} = \frac{1}{u} = \frac{1}{\log_e x}.$$

But $$\frac{du}{dx} = \frac{1}{x}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\log_e x} \cdot \frac{1}{x} = \frac{1}{x \log_e x}.$$

(2) $y = \log (x + 1 + \sqrt{2x + x^2})$. Find $\dfrac{dy}{dx}$.

Let $x + 1 + \sqrt{2x + x^2} = u$.

$$\therefore \quad \frac{du}{dx} = 1 + \frac{2 + 2x}{2\sqrt{2x + x^2}} = 1 + \frac{1 + x}{\sqrt{2x + x^2}} = \frac{\sqrt{2x + x^2} + 1 + x}{\sqrt{2x + x^2}}$$

and $$\frac{dy}{du} = \frac{1}{u}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{2x + x^2} + 1 + x}{\sqrt{2x + x^2}} \cdot \frac{1}{(x + 1) + \sqrt{2x + x^2}} = \frac{1}{\sqrt{2x + x^2}}.$$

(3) $y = (\log x)^n$. Find $\dfrac{dy}{dx}$.

Let $\log x = u$, $\therefore \; y = u^n$.

$$\therefore \quad \frac{dy}{du} = nu^{n-1} = n (\log x)^{n-1}.$$

$$\therefore \quad \frac{dy}{dx} \left(= \frac{dy}{du} \cdot \frac{du}{dx} \right) = \frac{n(\log x)^{n-1}}{x}.$$

(4) $y = e^{e^x}$. Find $\dfrac{dy}{dx}$.

If $e^x = u$, then $y = e^u$,

$$\therefore \quad \frac{dy}{du} = e^u = e^{e^x},$$

$$\therefore \quad \frac{dy}{dx} = e^{e^x} . e^x.$$

(5) $y = x^x$. Find $\frac{dy}{dx}$

$$\log y = x \log x,$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = x . \frac{1}{x} + 1 \times \log x = 1 + \log x,$$

$$\therefore \quad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x).$$

(6) $y = x^{x^{x^{\cdot^{\cdot^{\cdot \text{ to } \infty}}}}}$. Find $\frac{dy}{dx}$.

Here $y = x^y$, $\therefore$ $\log y = y \log x$,

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x},$$

$$\therefore \quad \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x},$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}\Big/\left(\frac{1}{y} - \log x\right),$$

$$= \frac{y^2}{x(1 - y\log x)}.$$

(7) Fechner's law states that the intensity of a sensation is proportional to the logarithm of the stimulus.

In symbolic form the law becomes

$$(y = k \log x),$$

where y is the perception and x is the stimulus.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{k}{x}\right),$$

i.e. the perceptibility of a sensation is inversely proportional to the stimulus.

(8) Cholera bacilli double themselves in number in 30 minutes. Find their rate of growth.

This being an example of growth in accordance with the compound interest law (see p. 94),

$$\therefore \quad 2 = e^{30k}, \text{ where } k \text{ is a constant.}$$

$$\therefore \quad 2{\cdot}303 \log_{10} 2 = 30k.$$

$$\therefore \quad k = \frac{2{\cdot}303 \times 0{\cdot}30103}{30} = 0{\cdot}0231.$$

$\therefore$ Law of growth is $y = e^{0{\cdot}0231t}$.

$\therefore$ Rate of growth $\frac{dy}{dt} = 0{\cdot}0231e^{0{\cdot}0231t}$.

(9) The following method of differentiation by taking logarithms first (to the base e) saves a lot of arithmetical labour in suitable cases (*i.e.* in cases of functions consisting of a number of factors).

$$y = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}}, \quad \therefore \quad \log y = \log x + \tfrac{1}{2}\log(1-x^2) - \tfrac{1}{2}\log(1+x^2).$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{x}{1-x^2} - \frac{x}{1+x^2} = \frac{1-2x^2-x^4}{x(1-x^2)(1+x^2)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}} \frac{(1-2x^2-x^4)}{x(1-x^2)(1+x^2)} = \frac{1-2x^2-x^4}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}.$$

(10) Prove that if $y = \sqrt{2x}$, then $\dfrac{dy}{dx} = \dfrac{1}{y}$.

$$\frac{dy}{dx} = \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}} = \frac{1}{y}.$$

(iii) Differentiation of Circular Functions.

(*a*) Direct Circular Functions.

The differential coefficient of sin x.

Let $y = \sin x$.

$$\therefore \quad y + dy = \sin(x+dx).$$

$$\therefore \quad dy = \sin(x+dx) - \sin x$$

$$= 2\sin\frac{dx}{2}\cos\frac{2x+dx}{2} \quad \text{(see p. 50).}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2\sin\dfrac{dx}{2}\cos\dfrac{2x+dx}{2}}{dx}.$$

But $\mathrm{Lt}_{dx\to 0}\dfrac{\sin dx}{dx} = 1$ (see p. 230), $\therefore \quad \dfrac{2\sin\dfrac{dx}{2}}{dx} = 1,$

and $\mathrm{Lt}_{dx\to 0}\cos\dfrac{2x+dx}{2} = \cos x.$

$$\therefore \quad \frac{dy}{dx}, \quad i.e. \ \frac{d\sin x}{dx}, = \cos x.$$

Similarly, $\dfrac{d\cos x}{dx} = -\sin x$

$$\frac{d\tan x}{dx} = \frac{d\dfrac{\sin x}{\cos x}}{dx} = \frac{\cos x \,.\, \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$= 1 + \tan^2 x.$$

and
$$\frac{d \sec x}{dx} = \frac{d\left(\dfrac{1}{\cos x}\right)}{dx},$$
$$= \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

The Peculiarity of sin θ and cos θ.—The student will have noticed that when $\sin\theta$ is differentiated with respect to θ the result is $\cos\theta$, and when $\cos\theta$ is differentiated with respect to θ it becomes $-\sin\theta$.

Similarly the differential coefficient of $\cos\theta$ is $-\sin\theta$ and the differential coefficient of $-\sin\theta$ is $-\cos\theta$. Hence we get the following *two curious results*, viz.:

(i) **Each of these functions when differentiated twice gives rise to the original function with the sign changed from + to −.**

(ii) **Each of these functions when differentiated four times gives rise to the original function with the original sign.**

These two trigonometrical ratios are the only functions which possess these peculiarities, and we shall see that great advantage is taken of these peculiarities for the purpose of expanding $\sin\theta$ and $\cos\theta$ in powers of θ (see p. 229). They are also useful to remember when solving differential equations of the type
$$\frac{d^2y}{dx^2} = -n^2y.$$

The student must bear in mind the fact that *these results are only true if the angle θ is expressed in circular measure* (i.e. *in radians*). In the calculus, as in all mathematical analyses, all angles are understood to be expressed in radians and not in degrees.

(*b*) **Inverse Circular Functions.**—We will find the differential coefficients of such functions as $\sin^{-1}\frac{x}{a}$, etc., from which the differential coefficients of such functions as $\sin^{-1}x$ will follow at once by taking $a = 1$.

If $\quad y = \sin^{-1}\frac{x}{a}, \quad$ then $\quad x = a \sin y.$

$$\therefore \quad \frac{dx}{dy} = a\cos y = a\sqrt{1-\sin^2 y} = a\sqrt{1-\frac{x^2}{a^2}} = \sqrt{a^2-x^2}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}. \quad \left[\text{Hence } \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}.\right]$$

Similarly, if

$$y = \cos^{-1}\frac{x}{a},\quad \frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}.\quad \left[\text{Hence}\,\frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}.\right]$$

Also, if $y = \tan^{-1}\dfrac{x}{a}$, then $x = a\tan y$.

$$\therefore\quad \frac{dx}{dy} = a\sec^2 y = a(1+\tan^2 y) = a\left(1+\frac{x^2}{a^2}\right) = \frac{a^2+x^2}{a}.$$

$$\therefore\quad \frac{dy}{dx} = \frac{a}{a^2+x^2}.\quad \left[\text{Hence}\,\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}.\right]$$

Similarly, if

$$y = \cot^{-1}\frac{x}{a},\quad \frac{dy}{dx} = -\frac{a}{a^2+x^2}.\quad \left[\text{Hence}\,\frac{d(\cot^{-1}x)}{dx} = -\frac{1}{1+x^2}.\right]$$

If $y = \sec^{-1}\dfrac{x}{a}$, $x = a\sec y$ $\left(\text{or } \cos y = \dfrac{a}{x}\right.$, and

$$\left.\sin y = \sqrt{1-\frac{a^2}{x^2}} = \frac{\sqrt{x^2-a^2}}{x}\right).$$

$$\therefore\quad \frac{dx}{dy} = a\sec y\tan y = x\tan y = \frac{x\sin y}{\cos y} = \frac{\sqrt{x^2-a^2}}{a/x}$$

$$= \frac{x\sqrt{x^2-a^2}}{a}.$$

$$\therefore\quad \frac{dy}{dx} = \frac{a}{x\sqrt{x^2-a^2}}.\quad \left[\text{Hence}\,\frac{d(\sec^{-1}x)}{dx} = \frac{1}{x\sqrt{x^2-1}}.\right]$$

Similarly, if

$$y = \operatorname{cosec}^{-1}\frac{x}{a},\quad \frac{dy}{dx} = -\frac{a}{x\sqrt{x^2-a^2}}.\quad \left[\text{Hence}\,\frac{d(\operatorname{cosec}^{-1}x)}{dx} = -\frac{1}{x\sqrt{x^2-1}}.\right]$$

EXAMPLES.

Find $\dfrac{dy}{dx}$ in the following cases:—

(1) $y = \sin nx$.

Put $nx = u$,

$$\therefore\quad \frac{dy}{du} = \cos u,$$

$$\therefore\quad \frac{dy}{dx} = \cos u\,.\,\frac{du}{dx} = n\cos u$$

$$= n\cos nx.$$

(2) $y = \sin^n x$.

Let $\sin x = u, \quad \therefore \quad y = u^n,$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = nu^{n-1} \cos x$$

$$= n \sin^{n-1} x \cos x.$$

(3) $y = \sin x^n$.

Let $x^n = u, \quad \therefore \quad y = \sin u,$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = nx^{n-1} \cos x^n.$$

(4) $y = \tan^3 x$.

Let $\tan x = u,$

$$\therefore \quad y = u^3,$$

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = 3 \tan^2 x \sec^2 x.$

(5) $y = e^{\sqrt{\sin x}}$.

Let $\sqrt{\sin x} = u,$

$$\therefore \quad y = e^u,$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^{\sqrt{\sin x}} \cdot \frac{\cos x}{2\sqrt{\sin x}}$$

(6) $y = \dfrac{\tan x}{x}$.

$$\frac{dy}{dx} = \frac{x \sec^2 x - \tan x}{x^2} = \frac{x - \frac{1}{2} \sin 2x}{x^2 \cos^2 x}$$

(7) $y = \sec x + \tan x$.

$$\frac{dy}{dx} = \sec x \tan x + \sec^2 x = \sec x (\tan x + \sec x).$$

(8) $y = \log_e (\sec x + \tan x)$.

Put $(\sec x + \tan x) = u,$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sec x + \tan x} \cdot \sec x (\tan x + \sec x)$$

$$= \sec x.$$

(9) $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x}}}$, etc. to infinity.

$$y^2 = \sin x + y,$$

$$\therefore \quad y^2 - y = \sin x,$$

$$\therefore \quad (2y - 1)\frac{dy}{dx} = \cos x,$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y - 1}.$$

But from $y^2 - y = \sin x$, we get $y = \dfrac{1 \pm \sqrt{1+4\sin x}}{2}$ (see p. 20)

$$\therefore \quad 2y - 1 = \pm\sqrt{1+4\sin x}, \quad \therefore \quad \frac{dy}{dx} = \pm\frac{\cos x}{\sqrt{1+4\sin x}}$$

(10) In Example (1), p. 47, when the hand is at A, the weight moves upwards with a velocity of 10 ins. per second. What is the angular velocity of the forearm at that instant?

Let distance AC $= x$.

Then angle AOC $(=\theta) = \sin^{-1}\dfrac{x}{12}$.

$$\therefore \quad \frac{d\theta}{dx} = \frac{1}{\sqrt{144-x^2}} \text{ (see p. 190).}$$

$$\therefore \qquad \text{Angular velocity} = \frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt}$$

$$= \frac{1}{\sqrt{144-x^2}}\cdot\frac{dx}{dt}$$

$$= \frac{10}{\sqrt{144-x^2}} \text{ radians per second}$$

$$\left(\text{since } \frac{dx}{dt} = 10 \text{ ins. per second}\right).$$

But since $\theta = 45°$ and AO $= 12$ ins.,

$$\therefore \quad x = AC = \frac{12\sqrt{2}}{2} = 6\sqrt{2},$$

$$\therefore \quad \frac{10}{\sqrt{144-x^2}} = \frac{10}{\sqrt{144-72}} = \frac{10}{\sqrt{72}} = \frac{5\sqrt{2}}{6} \text{ radians per second,}$$

$$= \frac{5\sqrt{2}}{6} \times 57{\cdot}3 \text{ degrees per second,}$$

$$= 67{\cdot}52° \text{ per second.}$$

(11) A galvanometer mirror M is 1 metre distant from the scale, and the spot of light is moving on the scale with a velocity of 15 cm. per second. When it is deflected 17 cm., what is the angular velocity of the beam of light at that instant, and what is the angular velocity of rotation of the mirror at that instant?

Let AB be the scale (the student is to draw the diagram for himself), M the mirror, and MH the distance of M from AB. Let S be the position of the spot of light on the scale at any instant.

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Then if we put $\quad HS = x$ and $\angle SMH = \theta,$

we have
$$\tan\theta = \frac{x}{MH} = \frac{x}{100},$$
$$\therefore \quad \theta = \tan^{-1}\frac{x}{100},$$
$$\therefore \quad \frac{d\theta}{dx} = \frac{1}{100}\frac{d\left(\tan^{-1}\frac{x}{100}\right)}{dx} = \frac{1}{100}\cdot\frac{1}{1+\frac{x^2}{10000}}$$
$$= \frac{100}{10000+x^2}.$$
$$\therefore \quad \frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{100}{10000+x^2}\cdot\frac{dx}{dt}.$$

But
$$\frac{dx}{dt} = 15 \text{ cm. per second},$$
$$\therefore \quad \frac{d\theta}{dt} = \frac{1500}{10000+x^2} \text{ radians per second.}$$

But
$$x = 17.$$
$$\therefore \quad \frac{d\theta}{dt} = \frac{1500}{10289} \text{ radians per second.}$$
$$= \frac{1500}{10289}\times 57{\cdot}3 \text{ degrees per second.}$$
$$= 8{\cdot}35^\circ \text{ per second.}$$

But beam of light rotates twice as quickly as the mirror,
$\therefore$ angular velocity of rotation of mirror $= 4{\cdot}18^\circ$ per second.

EXERCISES.

Differentiate the following:—

(1) $y = \sin 2x$. [*Answer*, $2\cos 2x$ (see Example (1)).]

(2) $y = \sin\sqrt{x}$. $\left[\textit{Answer}, \ \frac{\cos\sqrt{x}}{2\sqrt{x}} \text{ (see Example (3)).}\right]$

(3) $y = \dfrac{\cos 3x + \cos x}{\sin 3x - \sin x}$.

[*Answer*, $-\operatorname{cosec}^2 x$. Hint: Use identities in Chapter IV., p. 50. Expression becomes $= \cot x$; $\dfrac{d\cot x}{dx} = -\operatorname{cosec}^2 x$.]

(4) $y = \log\sin x$. $\left[\text{Let } \sin x = u; \ \frac{d\log_e u}{du} = \frac{1}{u} = \frac{1}{\sin x},\right.$
$$\left.\therefore \quad \frac{d\log_{10} u}{du} = \frac{0{\cdot}4343}{\sin x}, \quad \therefore \quad \frac{dy}{dx} = \frac{0{\cdot}4343}{\sin x}\cdot\cos x = 0{\cdot}4343\cot x.\right]$$

(5) $y = \log\cos x$.
$$\left[\text{If } u = \cos x, \text{ then } \frac{d\log u}{du} = \frac{0{\cdot}4343}{\cos x}, \text{ and } \frac{dy}{dx} = -0{\cdot}4343\tan x.\right]$$

(6) $y = \log \tan x$. $\left[\frac{dy}{dx} = \frac{0{\cdot}4343 \sec^2 x}{\tan x} = \frac{0{\cdot}4343}{\sin x \cos x} = \frac{0{\cdot}8686}{\sin 2x}.\right]$

(7) For what value of θ, between 0° and 180°, is $\tan\theta$ increasing four times as fast as θ?

$\left[\textit{Answer}, \frac{d \tan\theta}{d\theta} = \sec^2\theta = 4, \quad \therefore \quad \cos\theta = \pm\tfrac{1}{2}, \quad \therefore \quad \theta = 60° \text{ or } 120°.\right]$

(8) $y = \sqrt{\sin x\sqrt{\sin x\sqrt{\sin x \ldots}}}$

$\left[\textit{Answer}, \frac{dy}{dx} = \cos x \text{ (cf. Example (2), p. 22).}\right]$

List of Standard Differential Coefficients.

The following is a list of typical differential coefficients, familiarity with which will help the student to differentiate numerous other functions:—

Function, y.	*Differential Coefficient,* $\frac{dy}{dx}$.
x^m	mx^{m-1} (p. 162)
$\sin x$	$\cos x$ (p. 189)
$\cos x$	$-\sin x$ (p. 189)
$\tan x$	$\sec^2 x$ or $1+\tan^2 x$ (p. 189)
$\cot x$	$-\operatorname{cosec}^2 x$ or $-(\cot^2 x+1)$ (p. 194)
$\sec x$	$\sec x \tan x$ (p. 190)
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$ (p. 190)
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$ (p. 191)
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$ (p. 191)
$\cot^{-1}\frac{x}{a}$	$-\frac{a}{a^2+x^2}$ (p. 191)
$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$ (p. 191)
$\operatorname{cosec}^{-1}\frac{x}{a}$	$-\frac{a}{x\sqrt{x^2-a^2}}$ (p. 191)
$\log_e x$	$\frac{1}{x}$ (p. 187)

Function, y.	*Differential Coefficient,* $\frac{dy}{dx}$.
$\log_e (a \pm x)$	$\pm \frac{1}{(a \pm x)}$ (p. 187)
$\log_{10} (a \pm x)$	$\pm \frac{0{\cdot}4343}{(a \pm x)}$
$\log_e \sin x$	$\cot x$ (p. 194)
$\log_e \cos x$	$-\tan x$ (p. 194)
$\log_e \tan x$	$\frac{2}{\sin 2x} = 2 \operatorname{cosec} 2x$ (p. 195)
$\log_e \tan \frac{x}{2}$	$\operatorname{cosec} x$
$\log_e \tan \left(\frac{x}{2} + \frac{\pi}{4}\right)$	$\sec x$
e^x	e^x
e^{-x}	$-e^{-x}$
a^x	$\log_e a \,.\, a^x$

CHAPTER XI.

MAXIMA AND MINIMA.

It is often important to ascertain under what conditions some particular function of a given variable will have a maximum or minimum value, and what that maximum or minimum value may be. For example, it is found that reaction velocity under the influence of enzymes is very low at low temperatures and gradually rises as the temperature rises up to a certain limit, called the *optimum temperature*, but if the temperature is increased beyond that limit the velocity begins to diminish.

Again, we saw on p. 43, Chapter IV., that as the angle of pull of a muscle increases, the effective force of the muscle increases until the size of the angle reaches a certain value, when the force begins to diminish again.

In Chapter III. we dealt with algebraical methods of finding the maximum and minimum values of certain simple functions, but the differential calculus affords us an easy method of ascertaining the maximum and minimum values of all kinds of functions.

Maximum.—If a function of x increases in value while x is increased, and then begins to diminish when x is still further increased, the value of the function when the change occurs is called a maximum. Thus, in fig. 81, showing the graph of $y = 3x - x^2$, we see that at the point P, where $x = 1\frac{1}{2}$, y has a value equal to $2\frac{1}{4}$, and that at points on the curve close to and on either side of P, the ordinates are less than at P. At P, therefore, the value of y is a maximum.

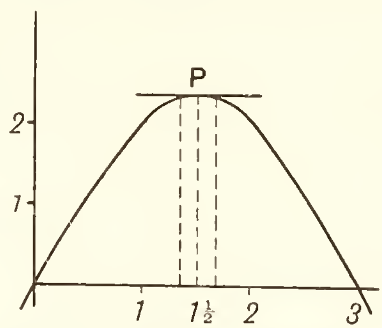


Fig. 81.—Graph of $y = 3x - x^2$.

Minimum.—If a function of x decreases in value while x is increased, and then begins to increase when x is still further increased, the value of the function when the change occurs is called a minimum. Thus, fig. 82 shows the graph of

$$y = x^2 - 3x + 3,$$

from which we see that at the point P, where $x = 1\frac{1}{2}$, y has a value equal to $\frac{3}{4}$, and that at the points on the curve close to and on either side of P the ordinates are greater than at P. At P, therefore, the value of y is a minimum.

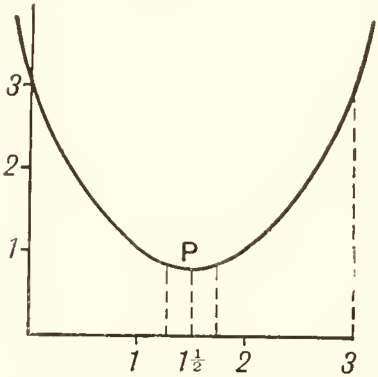


FIG. 82.—Graph of $y = x^2 - 3x + 3$.

It is important to remember that a **maximum** or a **minimum** ordinate is not always the greatest or smallest ordinate of the graph. For example, in fig. 83 the ordinate at P is a maximum ordinate, although the ordinate at A is greater. Similarly, the ordinate at S_1 is a minimum, although the ordinate at B is less. All that we mean by the statement that the ordinate y (or the function $y = f(x)$) has a maximum or minimum value at the points P and S_1, is that the ordinates at P or S_1 are respectively greater or less than ordinates close to and on either side of them. Indeed, one function may have several maxima and minima (*e.g.*, $y = \sin x$, $y = \cos x$ (see p. 127)) and, moreover, some of the minima may actually be greater than some of the maxima on the same curve, *e.g.* the minimum at S_1 (fig. 83) is greater than the maximum at P.

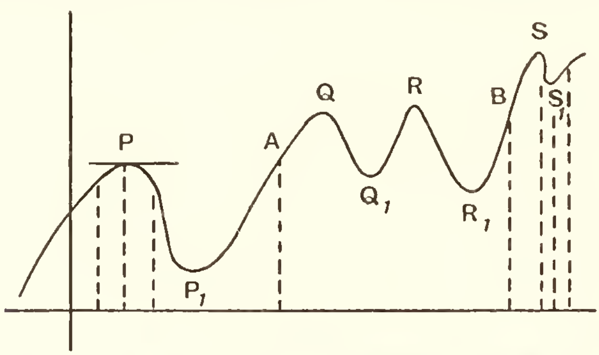


FIG. 83.—Diagram to illustrate Maxima and Minima.

Hence, the words maximum and minimum are not used mathematically with their ordinary meanings of "greatest possible" and "least possible."

Points of Inflection.—Another matter to remember is that although at all points of maximum and minimum the geometrical tangent is parallel to the x axis and therefore makes an angle

with that axis whose tangent = 0, so that $\frac{dy}{dx} = 0$, the reverse is not necessarily true, viz. we may have a point like Q (fig. 84) where the tangent is parallel to the x axis and where, therefore, $\frac{dy}{dx} = 0$, and yet the function has neither a maximum nor a minimum value at that point. Such a point is called a *point of inflection*; it is a point where the tangent crosses the curve, and the test for it is that $\frac{dy}{dx}$ does not change sign in passing through zero.

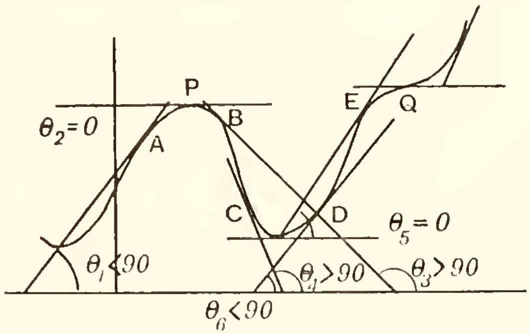


FIG. 84.—Point of Inflection at Q.

Investigation of Maximum and Minimum Values of a Function.—From figs. 84 and 85 it is obvious that at the maximum and

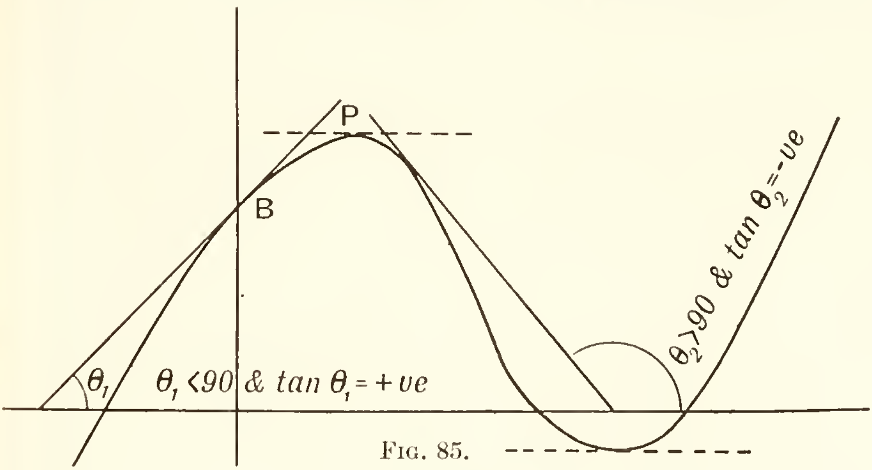


FIG. 85.

minimum points on a curve the tangent is parallel to the axis of x, *i.e.* it makes an angle 0° with the x axis. But the tangent of the angle which the slope of a curve at any point makes with the axis of x is, as we have seen on p. 161, represented by the value of $\frac{dy}{dx}$ at that point. Hence, we arrive at the following rule:—

To find the maximum or minimum value of any function

$y = f(x)$, differentiate the function, *i.e.* find the value of $\frac{dy}{dx}$, in the form of another function, and then equate this $\frac{dy}{dx}$ to zero. The value of x thus found gives the abscissa of the point whose ordinate is a maximum or a minimum.

An example will make this clear.

Let the function be $y = x^2 - 3x + 6$ (see fig. 86). Find the maximum or minimum value of this function, *i.e.* find for which value of x the function will be a maximum or a minimum.

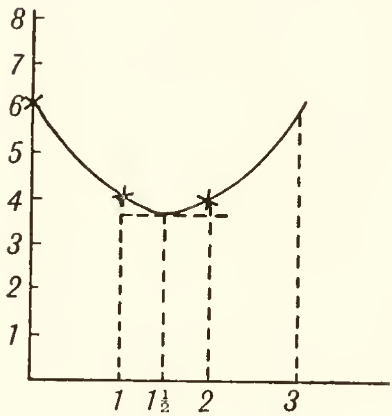


Fig. 86.—Graph of Function $y = x^2 - 3x + 6$.

Differentiating, we get

$$\frac{dy}{dx} = 2x - 3.$$

This $\frac{dy}{dx}$, as we have seen, represents the value of the angle which the tangent at any point on the curve makes with the axis of x.

$\therefore$ If for any value of x the curve has a maximum or minimum point, the tangent at that point will make an angle zero with the x axis, *i.e.* an angle whose tangent = 0.

$$\therefore \quad \text{at such a point } \frac{dy}{dx} = 0,$$

i.e. $\qquad 2x - 3 = 0,$

giving $\qquad x = \frac{3}{2}.$

Hence, the curve will have a maximum or a minimum at a point whose abscissa = $\frac{3}{2}$ (see fig. 86).

Discrimination between Maxima and Minima.—In order to ascertain whether the value found is a maximum or a minimum, there are several possible methods.

(1) **Trial Method.**—After finding the value of x for which $\frac{dy}{dx} = 0$, substitute this value in the function, and thus ascertain what value y assumes. Then give a slightly different value to x and see what effect such a new value of x has upon the value of y. The method is best illustrated by an example.

Take again the function

$$y = x^2 - 3x + 6.$$

We have seen that when $x = 1{\cdot}5, \quad \frac{dy}{dx} = 0.$

Now, when $\qquad x = 1{\cdot}5$
we have
$$\begin{aligned} y &= (1{\cdot}5)^2 - 3 \times 1{\cdot}5 + 6 \\ &= 3{\cdot}75. \end{aligned}$$

Let us give x the very slightly higher value 1·51.
We then have
$$\begin{aligned} y &= (1{\cdot}51)^2 - 3 \times 1{\cdot}51 + 6 \\ &= 3{\cdot}7501, \text{ which is greater than } 3{\cdot}75. \end{aligned}$$

Again, give x the slightly smaller value of 1·49.
We then have
$$\begin{aligned} y &= (1{\cdot}49)^2 - 3 \times 1{\cdot}49 + 6 \\ &= 3{\cdot}7501, \end{aligned}$$
which is again higher than 3·75.

Hence, we see that when $\frac{dy}{dx} = 0$ in this particular case, the value of y is a minimum.

(2) **The Method of Second Differentiation.**—From fig. 87 we

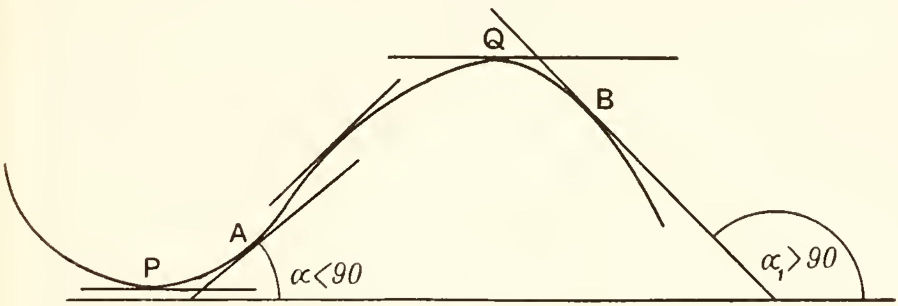


FIG. 87.—Diagram to illustrate the Method of Discriminating between a Maximum and a Minimum.

see that the slope of the curve continually changes, and that when we pass from a minimum value at P to the right, the angle of the slope changes from 0° at P to an angle less than 90° at A. But the tangent of an angle less than 90° is + ve, *therefore the slope changes from 0 to + ve when we pass from a minimum value to the right.* But when we pass from a maximum value at Q to the right, the angle of slope changes from 0 to an angle greater than 90° at B, and whose tangent is therefore − ve. *Hence the slope changes from 0 to − ve when we pass from a maximum to the right.* Now if we call the slope

of the curve at any point $\frac{dy}{dx}$, then $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ denotes the rate of change of slope.

But $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$, *i.e.* the second differential coefficient of y with respect to x. Therefore, *if $\frac{d^2y}{dx^2}$ is positive, then the value found by equating $\frac{dy}{dx}$ to zero is a minimum, but if $\frac{d^2y}{dx^2}$ is negative, then the value found by so equating $\frac{dy}{dx}$ is a maximum* (see p. 226).

Thus, in the case of $y = x^2 - 3x + 6$ (of page 200)

$$\frac{dy}{dx} = 2x - 3.$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2 = +\text{ve}.$$

$\therefore$ The value found is a minimum.

Let us take another example:

$$y = x^3 - 3x + 16.$$

Find the maxima and minima of this function.

$$\frac{dy}{dx} = 3x^2 - 3,$$

$$\frac{d^2y}{dx^2} = 6x.$$

Putting $3x^2 - 3 = 0$, we have $x^2 = 1$, $x = \pm 1$.

When $x = +1$, $6x$ is positive,

,, $x = -1$, $6x$ is negative.

Hence, $x = 1$ corresponds to a minimum, $y = 14$,

and $x = -1$,, ,, maximum, $y = 18$.

Trigonometrical Example.—Find the value of x which makes $\sin^3 x \cos x$ a maximum.

We have $$\frac{dy}{dx} = 3\sin^2 x \,.\, \cos^2 x - \sin^4 x = 0.$$

$$\therefore \quad \sin^2 x(3\cos^2 x - \sin^2 x) = 0.$$

When $\sin^2 x = 0$, the function $= 0$.

$\therefore$ the value $x = 0$ must be rejected.

$\therefore$ the function is a maximum when

$$\sin^2 x = 3\cos^2 x = 3(1 - \sin^2 x),$$

i.e. when $4\sin^2 x = 3$,

i.e. when $\sin x = \dfrac{\sqrt{3}}{2}$,

i.e. when $x = 60°$.

The value of the function then becomes

$$\frac{3\sqrt{3}}{8} \cdot \frac{1}{2} = \frac{3}{16}\sqrt{3}.$$

EXAMPLES.

(1) *Problem in Public Health.*—For an ellipse of given perimeter, find the relation between the major and minor axes so that the area may be a maximum. Hence show that the best shape to give to a water-pipe in order to prevent its bursting during frosty weather is that of an ellipse.

Whilst the perimeter of an ellipse cannot be accurately expressed in simple form, the formula $p = \pi(x+y)$ is an approximate expression for the perimeter, p, if x and y are the two semi-axes, and if x is nearly equal to y (*i.e.* if the ellipse is nearly circular).

From the formula $p = \pi(x+y)$, we have $y = \dfrac{p}{\pi} - x$.

But area A of an ellipse $= \pi xy$.

$$\therefore \quad A = \pi x\left(\frac{p}{\pi} - x\right) = px - \pi x^2.$$

$$\therefore \quad \frac{dA}{dx} = p - 2\pi x.$$

For a maximum $\left(\text{since } \dfrac{d^2A}{dx^2} \text{ is negative}\right)$ $p - 2\pi x = 0$,

whence $p = 2\pi x$

or
$$x = \frac{p}{2\pi} = \frac{\pi}{2\pi}(x+y) = \frac{x+y}{2}.$$

$$\therefore \quad 2x = x + y,$$

whence $x = y$.

$\therefore$ The given ellipse must have its semi-axes equal, *i.e.* it must be circular.

From this it follows that if a water-pipe is made slightly elliptical, then when the water inside it expands as a result of freezing, the pipe will tend to become more circular, for by so doing the area of its cross-section increases without altering its perimeter—thus preventing its bursting.

(2) *Another Problem in Public Health.*—A window is in the form of a

rectangle surmounted by a semicircle (fig. 88). If the perimeter is 30 feet, find the dimensions so that the greatest possible amount of light may be admitted.

From the diagram we see that total perimeter of window

$$= 2y + x + \frac{\pi x}{2}$$

$$= 30 \text{ feet (by hypothesis).}$$

$$\therefore \quad y = \frac{60 - x(2+\pi)}{4} \quad . \quad . \quad . \quad (1)$$

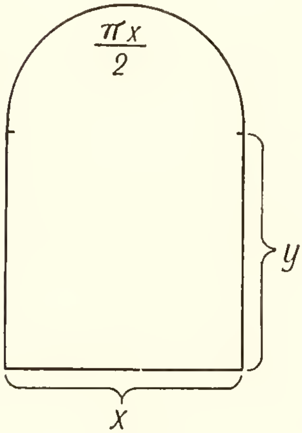


Fig. 88.

Now, area (A) of window = sum of areas of semicircular and rectangular parts

$$= \frac{\pi x^2}{8} + xy$$

$$= \frac{\pi x^2}{8} + \frac{x}{4}[60 - x(2+\pi)] \text{ (from (1)).}$$

$$\therefore \frac{dA}{dx} = 15 - \frac{x}{4}(4+\pi)$$

$$= 0 \text{ (for a maximum, since } \frac{d^2A}{dx^2} \text{ is negative).}$$

$$\therefore \quad x = \frac{60}{4+\pi} = 8{\cdot}4 \text{ feet.}$$

$$\therefore \quad y = \frac{60 - 8{\cdot}4(2+\pi)}{4} = 4{\cdot}2 \text{ feet.}$$

$$\therefore \quad \text{height of window} = y + \frac{x}{2} = 8{\cdot}4 \text{ feet.}$$

$$\therefore \quad \text{total height} = \text{width} = 8{\cdot}4 \text{ feet.}$$

(3) *Problem in Biochemistry.*—When ethyl acetate is hydrolysed in the presence of acetic acid as a catalyst the following reaction occurs:—

$$CH_3COOC_2H_5 + H_2O = CH_3COOH + C_2H_5OH.$$

The acetic acid formed thus gradually increases in amount and causes a uniform acceleration in the velocity of the reaction. But at the same time the active mass of the ester diminishes, thus causing a retardation in the reaction velocity. At what point will the velocity be a maximum?

Let the initial concentration of acetic acid $= a$ molecules per litre, and let the initial concentration of ester $= b$ molecules per litre.

Let x molecules be hydrolysed after a time t, producing x molecules of acetic acid.

$\therefore$ velocity due to acetic acid originally present,

$$= \frac{dx_1}{dt} = K_1 a(b-x).$$

Velocity due to acetic acid produced

$$= \frac{dx_2}{dt} = K_1 x(b-x).$$

$\therefore$ actual velocity $= \dfrac{dx_1}{dt} + \dfrac{dx_2}{dt}$

$= K_1(a+x)(b-x).$

$\therefore$ for a maximum, $\dfrac{d}{dx}(a+x)(b-x) = 0,$

i.e. $-(a-b)-2x = 0$, whence $x = \frac{1}{2}(b-a)$.

(4) *Problem in Physiology of Growth.*—On theoretical grounds it has been found by Robertson (see "*Child Physiology*," p. 249) that growth in the weight of infants up to nine months is an autocatalytic phenomenon taking place in accordance with the equation $\log \dfrac{x}{341{\cdot}5 - x} = K(t - 1{\cdot}66)$, where x is the weight of an infant in ounces at the age of t months.

At what age will the growth of the infant be most rapid?

Since $$\log \frac{x}{341{\cdot}5 - x} = K(t - 1{\cdot}66),$$

$$\frac{1}{K}\log \frac{x}{341{\cdot}5 - x} + 1{\cdot}66 = t.$$

Put $$\frac{x}{341{\cdot}5 - x} = z,$$

so that $$t = \frac{1}{K}\log z + 1{\cdot}66,$$

then $$\frac{dt}{dz} = \frac{1}{K} \cdot \frac{1}{z} = \frac{1}{K} \cdot \frac{341{\cdot}5 - x}{x},$$

and $$\frac{dz}{dx} = \frac{(341{\cdot}5 - x) \times 1 - x(-1)}{(341{\cdot}5 - x)^2}$$

$$= \frac{341{\cdot}5}{(341{\cdot}5 - x)^2}.$$

$$\therefore \quad \frac{dt}{dx} \text{ which} = \frac{dt}{dz} \cdot \frac{dz}{dx} = \frac{1}{K} \cdot \frac{341{\cdot}5 - x}{x} \cdot \frac{341{\cdot}5}{(341{\cdot}5 - x)^2}$$

$$= \frac{1}{K} \cdot \frac{341{\cdot}5}{x(341{\cdot}5 - x)},$$

$$= \frac{C}{x(341{\cdot}5 - x)}, \text{ where } C = \frac{341{\cdot}5}{K}.$$

$$\therefore \quad \text{Velocity of growth } v \text{ which} = \frac{dx}{dt} = \frac{1}{\frac{dt}{dx}}$$

$$= \frac{x(341{\cdot}5 - x)}{C}$$

$$= \frac{341{\cdot}5x - x^2}{C}$$

$$\therefore \quad \text{for a maximum } \left(\text{since } \frac{d^2v}{dx^2} \text{ is negative}\right) \frac{dv}{dx} = \frac{341{\cdot}5 - 2x}{C} = 0,$$

whence $$x = \frac{341{\cdot}5}{2} = 170{\cdot}75,$$

i.e. the growth of the infant is most rapid when the infant weighs 170·75 oz.

In order to find at what age the infant's weight is 170·75 oz. we return to the original equation:

$$\log \frac{x}{341 \cdot 5 - x} = \mathrm{K}(t - 1 \cdot 66).$$

$\therefore$ when $x = 170 \cdot 75$ oz., the left side of the equation becomes log I, which is 0,

$$\therefore \quad \mathrm{K}(t - 1 \cdot 66) = 0,$$

whence $t = 1 \cdot 66$ months = about seven weeks.

$\therefore$ the infant grows quickest at the age of seven weeks.

This result is confirmed by the table giving the weight of infants at various ages during the first nine months (see "*Child Physiology*," p. 251).

It will be noticed that the problem of finding when $\mathrm{K}x(341 \cdot 5 - x)$ is a maximum is the same as finding when $x(\mathrm{A} - x)$ is a maximum. This is the same as finding:

(i) How to divide a number A in such a way as to make the product a maximum. The answer is to divide it into two equal parts.

(ii) How to divide a line into two parts so as to make the rectangle contained by the two parts a maximum. The answer is to bisect the line.

(5) *Problem in General Physiology.*—Under what H-ion concentration will the sum of protein ions be a minimum in relation to the undissociated protein in solution?

As protein is an amphoteric electrolyte (*i.e.* it gives both H^+ and OH^- ions), there must be two forms of dissociation, viz.:

(*a*) $[A_p]$ and $[H^+]$, where A_p stands for protein anion.
(*b*) $[K_p]$ and $[OH^-]$, where K_p stands for protein kation.

Let the reaction velocity for the first form of dissociation be K_a, then, by the law of mass action (see p. 257),

$$K_a = \frac{[A_p] \cdot [H^+]}{[P]}$$

or

$$A_p = \frac{K_a \cdot [P]}{[H^+]}$$

where [P] stands for the concentration of non-dissociated protein.

If the reaction velocity for second form of dissociation be K_b,

then similarly $$K_p = \frac{K_b \cdot [P]}{[OH^-]}.$$

Hence $$\frac{[A_p] + [K_p]}{[P]} = \frac{K_a}{[H^+]} + \frac{K_b}{[OH^-]} = u \text{ (say)}.$$

Hence we have to find for what value of $[H^+]$, u will be a minimum.

We must therefore represent $\frac{K_a}{[H^+]} + \frac{K_b}{[OH^-]}$ as a function of $[H^+]$.

Now if K_w = dissociation constant of water, we have

$$[OH^-] \cdot [H^+] = K_w,$$

whence $$[OH^-] = \frac{K_w}{[H^+]}.$$

$$\therefore \quad \frac{K_a}{[H^+]} + \frac{K_b}{[OH^-]} = \frac{K_a}{[H^+]} + \frac{K_b[H^+]}{K_w} = u.$$

Hence for a minimum we must have

$$\frac{du}{d[\mathrm{H}^+]} \text{ which } = -\frac{\mathrm{K}_a}{[\mathrm{H}^+]^2}+\frac{\mathrm{K}_b}{\mathrm{K}_w} = 0,$$

or

$$\frac{\mathrm{K}_a}{[\mathrm{H}^+]^2} = \frac{\mathrm{K}_b}{\mathrm{K}_w},$$

whence

$$\frac{\mathrm{K}_a}{\mathrm{K}_b} = \frac{[\mathrm{H}^+]^2}{\mathrm{K}_w} = \frac{[\mathrm{H}^+]^2}{[\mathrm{H}^+]\,.\,[\mathrm{OH}^-]} = \frac{[\mathrm{H}^+]}{[\mathrm{OH}^-]}.$$

Hence there will be a minimum of dissociated protein ions when the H^+ and OH^- ion concentrations are in the same relation as the corresponding dissociation constants or reaction velocities, *i.e.* at the **iselectric** point.

In order to find the value of $[H^+]$ under those conditions we proceed as follows:—

$$[\mathrm{H}^+] = \frac{\mathrm{K}_a}{\mathrm{K}_b}[\mathrm{OH}^-].$$

But

$$[\mathrm{H}^+]\,.\,[\mathrm{OH}^-] = \mathrm{K}_w = 10^{-14}$$

(this being the dissociation constant of pure water).

$$\therefore \quad [\mathrm{OH}^-] = \frac{10^{-14}}{[\mathrm{H}^+]}.$$

$$\therefore \quad [\mathrm{H}^+] = \frac{\mathrm{K}_a}{\mathrm{K}_b}\frac{10^{-14}}{[\mathrm{H}^+]}.$$

$$\therefore \quad 10^{-14}\frac{\mathrm{K}_a}{\mathrm{K}_b} = [\mathrm{H}^+]^2.$$

$$\therefore \quad [\mathrm{H}^+] = 10^{-7}\sqrt{\frac{\mathrm{K}_a}{\mathrm{K}_b}}.$$

That this value of $[H^+]$ gives a minimum and not a maximum is seen from the fact that $\frac{d^2u}{d[\mathrm{H}^+]^2} = \frac{2\mathrm{K}_a}{[\mathrm{H}^+]^3} = +\text{ve}.$

(6) *Problem in Morphology of Blood Vessels.*—John Hunter wrote as follows: "To keep up a circulation **sufficient for the part and no more,** Nature has varied the angle of origin of the arteries accordingly." Prove the truth of this statement, on the assumption that the loss of pressure between any point A on the main trunk and a point D on the branch artery BD is mainly due to friction of the blood-stream against the arterial walls, and is therefore, in accordance with Hess's law, proportional to

$$\frac{\mathrm{AB}}{\mathrm{R}}+\frac{\mathrm{BD}}{r}$$

where R and r are the radii of the vessels.

The route by which the blood could be conveyed from A to D can be any one of the following (fig. 89). Either—

(1) By a branch from A to D inclined to the main trunk at an angle θ_1; or
(2) By a branch from C to D at right angles to the main trunk; or
(3) By a branch from any point B (between A and C) to D, making an acute angle θ with the main trunk; or
(4) By a branch from any point E (beyond C) to D, making an obtuse angle θ_2 with the main trunk.

The problem which remains to be solved therefore, is, along which route will the loss of pressure between A and D be a minimum?

Let us take any point B on the main trunk. Then loss of pressure between A and D is according to Hess's law proportional to

$$\frac{AB}{R} + \frac{BD}{r} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, D being a fixed point on the branch vessel, let its perpendicular distance DC from the main trunk be h.

Also let distance of B from C be x (*i.e.* the unknown point at which the branch vessel comes off is x units distant from C).

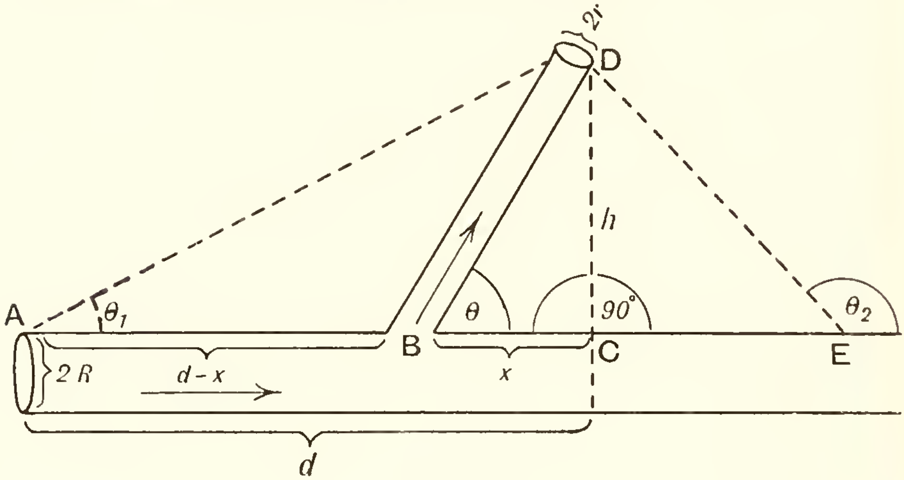


FIG. 89.

Also let distance AC = d.
We then have the following:—

$$AB = d - x,$$
$$BD = \sqrt{h^2 + x^2}$$

(BD being the hypotenuse of a right-angled triangle).

∴ By (1), fall of pressure between A and D is proportional to

$$\frac{d - x}{R} + \frac{\sqrt{h^2 + x^2}}{r},$$

∴ the route must be such that

$$\frac{d - x}{R} + \frac{\sqrt{h^2 + x^2}}{r}$$

is a minimum.

Calling this function y, we must have

$$\frac{dy}{dx} = 0 \text{ (for a minimum, since } \frac{d^2y}{dx^2} \text{ is positive),}$$

i.e.
$$-\frac{1}{R} + \frac{2x}{2r\sqrt{x^2 + h^2}} = 0,$$

whence
$$\frac{r}{R} = \frac{x}{\sqrt{x^2 + h^2}} = \frac{x}{BD} = \cos\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$\therefore$ The angle at which the vessel comes off is such that its cosine is $\frac{r}{R}$.

To find the distance x, we have

$$\frac{r}{R} = \frac{x}{\sqrt{x^2+h^2}}.$$

$$\therefore \quad \frac{x^2}{x^2+h^2} = \frac{r^2}{R^2},$$

i.e. $$x^2(R^2-r^2) = h^2r^2,$$

whence $$x = \frac{hr}{\sqrt{R^2-r^2}} \quad . \quad . \quad . \quad . \qquad (3)$$

From (2) we learn that

(i) All branch vessels of the same radius coming off from the same trunk will make the same angle with the main trunk;

(ii) If $\frac{r}{R}$ is very small, then $\cos\theta$ is very small, *i.e.* a branch of very small calibre comes off practically at a right angle (viz. at C);

(iii) If $\frac{r}{R}$ is nearly unity, then $\theta = 0$, *i.e.* a branch of very large calibre comes off practically parallel to the main trunk (*e.g.* the external and internal carotids). (See D'Arcy W. Thompson's "*Growth and Form*," and Burns' "*Biophysics*.")

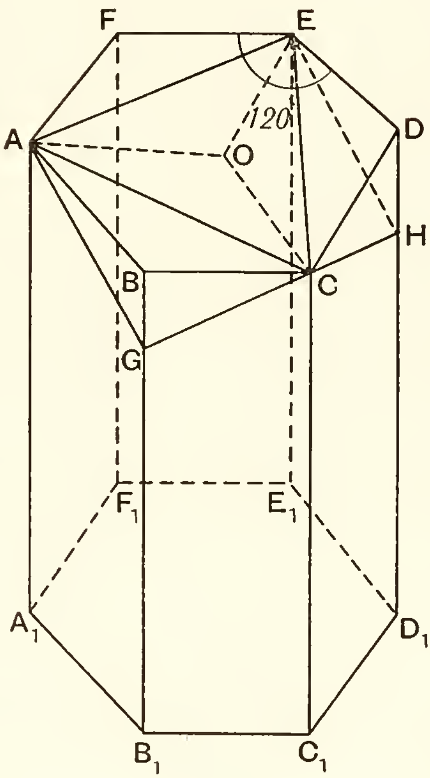


FIG. 90.

(7) *The Economy of the Bee.*—A bee cell may be considered as a regular hexagonal prism modified in the following manner (J. Salpeter):—

Let ABCDEF be a base of the prism (fig. 90). Join AC and let a plane ACG inclined at an angle a to the base ABCDEF cut the border BB_1 in the point G.

Rotate the pyramid ABCG round AC until the triangle ABC comes to lie in the position AOC. The apex G will then project upwards. Pass similar planes through CE and AE, also making angles a with the base ABCDEF, and rotate the pyramids until the triangles CDE and AFE coincide with the triangles COE and AOE respectively.

The three little pyramids will then meet in a common apex S (fig. 91). **The resulting figure is then similar to a bee cell.**

It is clear that by turning the pyramid ABCG and the other two pyramids round the lines AC, etc., the *volume* of the cell has not been diminished, but the *area* has been diminished, since although the three rhombi (AGCS

and the other two analogous rhombi) have been added to the area, yet there has been removed the whole of the surface ABCDEF, and the surfaces of the six triangles ABG, CBG, and the other four analogous triangles. Let us therefore consider **what must be the angle a between each of the planes and the base in order to make the area of the resulting cell a minimum.** When this occurs the bees will obviously have to use less wax in order to build a cell containing a given amount of honey.

In order to solve this problem it will be better to consider what must be

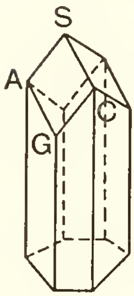


FIG. 91.

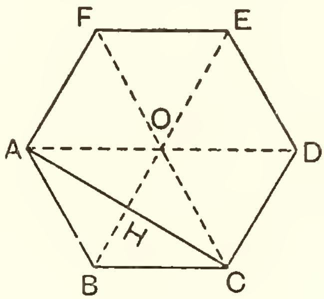


FIG. 92.

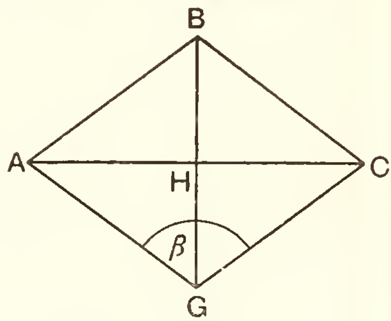


FIG. 93.

the length BG (fig. 90), since this length also determines the shape of the cell.

Let this length BG $= x$.

Now if we have a hexagon (fig. 92), and we divide it by means of the three diagonals into six equilateral triangles, OAB, OBC, etc., then the height h (*e.g.* AH) of each triangle is given by

$$h = \sqrt{\text{AB}^2 - \text{BH}^2} = \sqrt{a^2 - \frac{a^2}{4}} = \frac{a\sqrt{3}}{2}$$

(where a = length of side of hexagon).

$$\therefore \quad \text{AC} = 2h = a\sqrt{3}.$$

$$\begin{aligned} \therefore \quad \text{Surface of hexagon} &= 3 \text{ times surface ABCO} \\ &= 6 \text{ times surface ABC} \\ &= 6\frac{\text{AC}}{2} \,.\, \text{BH} = 6\frac{a\sqrt{3}}{2} \,.\, \frac{a}{2} = \frac{3a^2\sqrt{3}}{2}. \end{aligned}$$

Again (in fig. 90) $\quad \text{GC}^2 = \text{BC}^2 + x^2 = a^2 + x^2,$

$$\therefore \quad \text{GC} = \sqrt{a^2 + x^2}.$$

Now, surface of rhombus AGCS (fig. 91) or AGCB (fig. 93)

$$\begin{aligned} &= \text{twice the surface of the triangle AGC} \\ &= \text{AC} \,.\, \text{GH}. \end{aligned}$$

But

$$\begin{aligned} \text{GH} &= \sqrt{\text{GC}^2 - \text{HC}^2}, \\ &= \sqrt{(a^2 + x^2) - \frac{3a^2}{4}} \\ &\qquad \left(\text{since HC} = \frac{\text{AC}}{2} \text{ and AC} = a\sqrt{3}\right) \\ &= \sqrt{\frac{a^2}{4} + x^2}. \end{aligned}$$

$\therefore$ Surface of rhombus (which $=$ AC . GH)

$$= a\sqrt{3} \cdot \sqrt{\frac{a^2}{4}+x^2}.$$

To form the bee's cell, one has to take away from the surface of the prism—

(i) The surface of the hexagonal base,

i.e.
$$\frac{3a^2}{2}\sqrt{3}.$$

(ii) The surfaces of six triangles like ABG,

i.e.
$$6 \cdot \frac{ax}{2} = 3ax.$$

On the other hand one has to add—

(i) The three rhombi like AGCS (fig. 91),

$$= 3a\sqrt{3}\sqrt{\frac{a^2}{4}+x^2}.$$

$\therefore$ The total diminution (y) in surface is given by

$$y = \frac{3a^2\sqrt{3}}{2} + 3ax - 3a\sqrt{3}\sqrt{\frac{a^2}{4}+x^2}.$$

Now for the area of the resulting cell to be a minimum (for a given volume) the portion y that is taken away must of course be a maximum. Hence we must find the value of x which will make y a maximum.

Differentiating and equating to zero we get

$$\frac{dy}{dx} = 3a - 3a\sqrt{3} \cdot \frac{2x}{2\sqrt{\frac{a^2}{4}+x^2}} = 0,$$

i.e.
$$1 = \frac{x\sqrt{3}}{\sqrt{\frac{a^2}{4}+x^2}},$$

or
$$\frac{a^2}{4}+x^2 = 3x^2,$$

whence
$$2x^2 = \frac{a^2}{4},$$

giving
$$x = \frac{a\sqrt{2}}{4}.$$

To show that this value of x makes the surface of the *portion to be removed* a maximum and not a minimum, differentiate again and we have

$$\frac{d^2y}{dx^2} = -\text{ve},$$

$\therefore$ $x = \frac{a\sqrt{2}}{4}$ makes the surface of the *resulting* cell a minimum.

To find the angle β (*i.e.* angle AGC) of the rhombus AGCS (fig. 91) or AGCB (fig. 93) under such conditions, we have

$$\sin\frac{\beta}{2} = \frac{\text{AH}}{\text{AG}} = \frac{a\sqrt{3}}{2} \div \sqrt{\left(\frac{a}{4}\sqrt{2}\right)^2 + a^2}$$

$$= \tfrac{1}{3}\sqrt{6} = 0{\cdot}8165 = \sin 54° \ 44' \ 7'',$$

whence $$\beta = 109° \ 28' \ 14''.$$

This value agrees very closely with the value of the angle as found by actual measurement.

There is an interesting story associated with this angle β. It is narrated that Maraldi measured the angle and found it to be 109° 28′, and later, at the instigation of Réaumur, König calculated the value of β as 109° 26′. Maclaurin, dissatisfied with the discrepancy of 2′ between the calculated and observed results, repeated Maraldi's measurements and found them correct. He then discovered that König's logarithm tables were not absolutely correct, so that the bee cell has served to discover a mistake in logarithmic tables.

This story is very interesting and pretty, but unfortunately can only be regarded as an anecdote and nothing more, since recent measurements undertaken by Vogt in the case of 4000 bee cells gave the average value of β as 107° !

What the cause of such a discrepancy between the calculated and measured results is, it is impossible to say. Possibly there may be factors operating other than those of surface tension—to which latter is to be ascribed the tendency for the production of minimal surfaces (see "*Child Physiology*," p. 94). Perhaps the discrepancy may help to bring about further discoveries in the same way as the observed perturbation in the calculated orbit of Uranus helped to bring about the discovery of Neptune.

(8) *Problem in Neurophysiology.*—The ratio between the radius of the axon and that of the myelin sheath of a nerve has been found to be 1 : 1·6. Assuming that the myelin coat has an insulating function like the covering of a submarine telegraph cable, is this value of the ratio such as to make the velocity of a nerve impulse a maximum, it being known that the speed of signalling along a submarine cable varies as $x^2 \log_e \frac{1}{x}$, where x is the ratio between the radius of the core and that of the covering.

If $$y = \text{velocity of impulse},$$

then $$y = \text{K}x^2 \log_e \frac{1}{x} = -\text{K}x^2 \log_e x,$$

$$\therefore \quad \frac{dy}{dx} = -\text{K}\left(2x \log_e x + x^2 \cdot \frac{1}{x}\right),$$

$$= -\text{K}x(2\log_e x + 1);$$

$\therefore$ For a maximum, $\log x = -\tfrac{1}{2}$,

$$\therefore \quad x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} = 1 : 1{\cdot}65.$$

Hence, on the given assumption, the observed and calculated results are in agreement.

That $x = 1 : 1{\cdot}65$ gives a maximum and not a minimum speed is seen from a second differentiation:

$$\frac{d^2y}{dx^2} = -\text{K} \,.\, 2\log_e x - 2\text{K}x \,.\, \frac{1}{x} - \text{K}$$
$$= -\text{K}(2\log_e x + 3) = -\text{ve}.$$
$$\therefore \quad x = 1 : 1{\cdot}65 \text{ gives a maximum.}$$

(See W. M. Feldman, *Proc. Physiol. Soc.*, 1923.)

(9) Two electric light bulbs of 32- and 16-candle-power respectively are placed 10 feet apart on a table. The intensity of illumination being inversely proportional to the square of the distance, which point on the table between the lamps is best lighted?

Let the point be x feet from the lamp of 32-candle-power. Therefore its distance from the 16-candle-power lamp is $(10-x)$ feet.

$\therefore$ Intensity of illumination at that point is proportional to

$$\frac{32}{x^2} + \frac{16}{(10-x)^2} \quad \text{or to} \quad \frac{2}{x^2} + \frac{1}{(10-x)^2}.$$

For a maximum we must have by differentiation:

$$\frac{2}{x^3} - \frac{1}{(10-x)^3} = 0, \quad \text{or} \quad \frac{\sqrt[3]{2}}{x} = \frac{1}{(10-x)},$$
$$\therefore \quad (10-x)\sqrt[3]{2} = x,$$

whence

$$x = \frac{10\sqrt[3]{2}}{1+\sqrt[3]{2}} = \frac{12{\cdot}6}{2{\cdot}26} = 5 \text{ feet } 6{\cdot}9 \text{ inches (from the 32-candle-power lamp).}$$

(10) A photographic lens of 10 inches focal length forms an image of an object. What is the minimum possible distance between the object and the image?

$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$ (where f = focal length, and v and u are the distances of image and object respectively from lens).

$$\therefore \quad \frac{1}{10} = \frac{1}{v} + \frac{1}{u}.$$
$$\therefore \quad \frac{1}{v} = \frac{1}{10} - \frac{1}{u} = \frac{u-10}{10u},$$

whence

$$v = \frac{10u}{u-10}.$$
$$\therefore \quad v+u = \frac{10u}{u-10} + u = \frac{u^2}{u-10}.$$

$$\therefore \quad \text{For a minimum} \quad \frac{d\left(\frac{u^2}{u-10}\right)}{du} = 0, \ i.e. \ 2u(u-10) - u^2 = 0,$$

whence $\quad u = 20$ inches and $v = \dfrac{10u}{u-10} = 20$ inches.

$\therefore$ Minimum value of $v+u = 40$ inches.

(11) A book lying on a table is 5 feet distant from a wall. At what height above the book should a light be fixed in the wall so as to afford

maximum illumination to the book, it being given that the intensity of light varies directly as the cosine of the angle of incidence and inversely as the square of the distance.

In the diagram (fig. 94) we have

$$y \text{ (intensity of illumination)} = \frac{\cos a}{AB^2} = \frac{\cos a}{x^2+25}$$
$$= \frac{x/\sqrt{x^2+25}}{(x^2+25)}$$
$$= x(x^2+25)^{-\frac{3}{2}}.$$

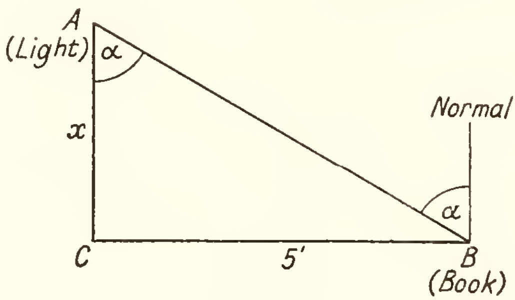


FIG. 94.

$\therefore$ For a maximum $\frac{dy}{dx} = (x^2+25)^{-\frac{3}{2}} - \frac{3}{2} \cdot 2x(x^2+25)^{-\frac{5}{2}} \cdot x = 0.$

$$\therefore \quad 1 - 3x^2(x^2+25)^{-1} = 0,$$
$$\therefore \quad x^2+25-3x^2 = 0,$$
$$\therefore \quad x = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$
$$= 3{\cdot}535.$$

$\therefore$ Light must be placed 3·535 feet above the book.

EXERCISES.

(1) An open tank is to be constructed with a square base and vertical sides, so as to contain a given quantity of water. What must be the relation between depth and width so as to make the expense of lining it with lead a minimum?

[*Answer*, Depth $= \frac{1}{2}$ width.]

(2) If $y = x^3 - 6x^2 + 11x - 6$, find the values of x for which y will be a maximum or a minimum.

[*Answer*, For maximum $x = 2 - \frac{\sqrt{3}}{3}$,

For minimum $x = 2 + \frac{\sqrt{3}}{3}$.]

(3) If $y = \frac{\log_e x}{x}$, find the value of x for which y will be a maximum.

[*Answer*, For a maximum $\log_e x = 1$, $\therefore$ $x = e$ and $y = e^{-1}$.]

(4) If $y = x^{\frac{1}{x}}$, prove that the minimum value of y is when $x = e$.

$$\left[Answer, \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}(\log x - 1), \quad \therefore \quad \text{for minimum}, \log x = 1.\right]$$

(5) What number exceeds its square by the greatest number possible?

[*Answer*, $x - x^2 = \text{max.}, \quad \therefore \quad 1 - 2x = 0$, whence $x = \frac{1}{2}$.]

(6) What number exceeds its cube by the greatest number possible?

[*Answer*, $x - x^3 = \text{max.}, \quad \therefore \quad 1 - 3x^2 = 0, \quad \therefore \quad x = 0{\cdot}5774$.]

(7) A piece of wire 5 feet long is cut into two parts. One part is bent to form a square and the other to form a circle. What are the lengths of the parts when the sum of the areas formed is a minimum?

$$\left[Answer, \left(\frac{5-x}{4}\right)^2 + \frac{x^2}{4\pi} = \text{min.},\right.$$

$$\therefore \quad \frac{-10+2x}{16} + \frac{x}{2\pi} = 0,$$

$$\left.\therefore \quad x = 2{\cdot}2 \text{ feet (for circle) and } 5 - x = 2{\cdot}8 \text{ feet (for square).}\right]$$

Curvature.—By the curvature of a circle is meant the rate at which the circumference curves round. Thus, if we look at the various circles in the diagram (fig. 95) we see that in the

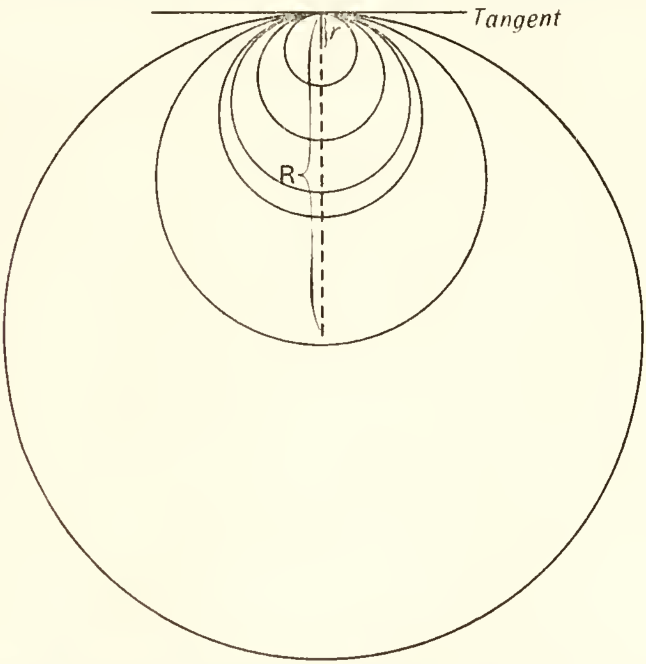


Fig. 95.—Diagram to illustrate the Meaning of the Term Curvature.

largest circle the rate of curving is least and in the smallest circle it is most. Hence we see that the curvature of a circle is measured by the reciprocal of its radius. Thus, if the radii

of the smallest and largest circles be r and R respectively, then the curvature of the smallest circle $= \frac{1}{r}$ and the curvature of the largest circle $= \frac{1}{R}$.

Radius of Curvature.—Let DE be any curve and A, B, C three points on it very close together (fig. 96). Then a circle ABCF can be drawn through these three points, and it will be seen that as the distance between the three points becomes less and less, until A and C ultimately coincide with B, the circle passing through these points will approach and finally attain the same curvature as the curve at the point B.

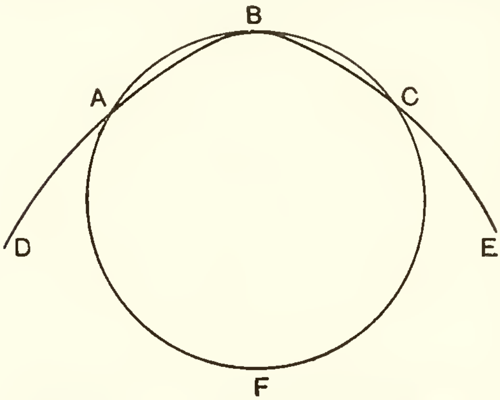


Fig. 96.—Diagram to illustrate the Meaning of the Term "Radius of Curvature."

Hence, *the radius of curvature* at any point B of a given curve is the radius of the circle which has the same curvature as the curve at that point.

Formula for Curvature.—It can be shown that,

if ρ = radius of curvature of curve $y = f(x)$,

then curvature, which $= \frac{1}{\rho}$, is given by the formula

$$\frac{1}{\rho} = \frac{y_2}{(1+y_1^2)^{\frac{3}{2}}},$$

where $$y_1 = \frac{dy}{dx},$$

and $$y_2 = \frac{d^2y}{dx^2},$$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}.$$

Example.—Find the radius of curvature of the parabola $y = 2x^2$ at the points where $x = 0$ and $x = \frac{1}{3}$.

From the formula

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

we have, since $y_1 = 4x$ and $y_2 = 4$,

$$\rho = \frac{(1+16x^2)^{\frac{3}{2}}}{4};$$

$\therefore$ where $x = 0$, $\rho = \frac{1}{4}$,

and where $x = \frac{1}{3}$, $\rho = \frac{(1+\frac{16}{9})^{\frac{3}{2}}}{4} = \left(\frac{25}{9}\right)^{\frac{3}{2}}\Big/4$

$$= \left(\frac{5}{3}\right)^3\Big/4$$

$$= \frac{125}{108}.$$

Relation between Tension and Radius of Curvature.

If we stretch an elastic band between two points on a flat surface it will obviously exercise no pressure at all on any part of the surface below. But if the band is stretched over a curved surface (*e.g.* a cylinder), then the downward pressure of the band will be proportional to the curvature (or inversely proportional to the radius of curvature) of the surface. Hence,

if p = downward pressure of band,
T = tension with which band is stretched,
R = radius of curvature of surface,

then $p = \frac{T}{R}$ (per unit of surface).

If instead of a cylinder which is curved only in one direction, the band be stretched over a surface which has curvatures in two directions, then

$$p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where R_1 and R_2 are the two radii of curvature.

Further, if $R_1 = R_2 = R$ (*e.g.* in the case of a sphere),

then $$p = \frac{2T}{R},$$

$$\therefore \quad T = p\frac{R}{2}.$$

But the thickness of the wall of a hollow vessel must be proportional to the tension to which the vessel is subjected; therefore, when p is constant, the thickness is proportional to the radius of curvature. Hence, *in the case of the uterus, the thickness of muscle is greater at the fundus*, where the radius of curvature is greater, *than at the cervix*, where the radius of curvature is less. Also the hemispherical aortic valves need have only half the thickness of the cylindrical aorta.

Similarly, *the cardiac apex*, which has the greatest curvature (or least radius of curvature), *is the thinnest part of the heart.* The same is the case with blood-vessels. Other illustrations are the constipation associated with intestinal distension (*e.g.* megalocolon) because, on account of the larger radius of curvature, a larger force is required to drive the contents forward, and the acceleration of labour after the rupture of the membranes, because the diminution in radius of curvature enables the same tension in the uterine wall to exert a higher pressure on the contents.

EXAMPLES.

(1) The lumen of the sheep's carotid is 3 mm.; that of the ox's carotid is 6 mm. The blood pressure in these vessels has been found to be 40 mm. in the case of the sheep, and 60 mm. in the case of the ox. If the thickness of the walls of the sheep's carotid is 0·616 cm., what would you expect to be the thickness of the coats of the carotid in the ox?

The radii of curvature of the sheep's and ox's carotids are in the proportion of 3 : 6 or 1 : 2.

The pressures in the vessels in these cases are in the proportion of 40 : 60 or 2 : 3.

Now, thickness (t) (which is proportional to the tension) is proportional to the pressure and radius of curvature,

$$\therefore \quad t_{(\text{sheep})} = \text{KR}_s\text{P}_s \text{ (where } \text{R}_s = \text{radius of sheep's carotid,}$$
$$\text{P}_s = \text{pressure in sheep's carotid,}$$
$$\text{and K} = \text{constant),}$$

and $$t_{(\text{ox})} = \text{KR}_o\text{P}_o \text{ (}o\text{ standing for ox);}$$

$$\therefore \quad \frac{t_{(\text{sheep})}}{t_{(\text{ox})}} = \frac{\text{R}_s\text{P}_s}{\text{R}_o\text{P}_o} = \frac{1 \,.\, 2}{2 \,.\, 3}$$
$$= 1/3.$$

$\therefore$ Thickness of ox's carotid should be three times that of sheep's carotid,
$$= 3 \times 0{\cdot}616 = 1{\cdot}848 \text{ cm.}$$
(Actual observation shows $t_{(\text{ox})}$ to be 1·744.)

(2) If the radius of a capillary is 0·000005 cm., find the amount of intracapillary tension that will maintain a difference of pressure of 50 mm. of mercury between the inside and the outside of the capillary.

A pressure of 50 mm. mercury = 5 × 13·6 = 68 grammes per sq. cm.

$$\therefore \quad 68 = \frac{\text{T}}{0{\cdot}000005}, \text{ giving T} = 0{\cdot}34 \text{ mgm. per cm. length.}$$

[See Cranston Walker, *Br. Med. Journ.*, 18th February 1922; and Correspondence by Gillespie, McQueen, Leonard Hill and others, *ibid.*, 1921.]

CHAPTER XII.

ESTIMATION OF ERRORS OF OBSERVATION.

In the first and second chapters reference was made to the subject of accuracy and limits of error in calculation. In the present chapter it will be shown how differentiation enables us to estimate the degree of accuracy of the value of the dependent variable y in any function $y=f(x, z, t, \ldots)$, when the degree of accuracy of the values of all of the independent variables is known. The problem before us may be stated broadly as follows:—

We have, on theoretical grounds, established a relationship between several variables in the form $y=f(x, z, t, \ldots)$, *e.g.* $y=x^3$, $y=\sqrt{x}$, $y=\sqrt{x^2+z^2}$, $y=xz$, $y=x \sin t$, etc. The various values of x, z, t, etc. in the function have been obtained as the result of laboratory observations and measurements which we know to be liable to errors reaching a certain limit of, say, p per cent. These errors will, naturally, affect the accuracy of the value y. How are we to estimate the effect of these errors on the value of y? Such an estimate is important because if we find the value of y to lie outside the estimated limits of error, the divergence between the calculated and observed values cannot be due to an error of observation, and we must therefore conclude that either the supposed theoretical relationship between the variables does not hold true, or that some unaccounted factor was present to vitiate the result (see p. 349).

We shall consider the subject under two headings, as follows:—

(1) One source of error, *e.g.* in such cases as $y=f(x)$.

(2) More than one source of error, *e.g.* in such cases as $y=f(x, z, t, \ldots)$.

(1) **Cases of $y = f(x)$.**—Let Δx be the limit of error in the measurement of x, and Δy the corresponding limit of error in the value of y.

Then, if Δx and, therefore, also Δy are very small, we have

$$\frac{\Delta y}{\Delta x}=\frac{dy}{dx}, \text{ approximately,}$$

$$\therefore \quad \Delta y = \frac{dy}{dx} \cdot \Delta x, \text{ approximately,}$$

and the percentage error in y is

$$\frac{100\Delta y}{y} = \frac{100}{y} \cdot \frac{dy}{dx} \cdot \Delta x, \text{ approximately.}$$

EXAMPLES.

(1) The side of a cube was measured and found to be 7·35 cm. long with a possible limit of error of 2 per cent. What is the limit in the value of (*a*) the area of one of the faces, (*b*) the volume of the cube.

(*a*) Area, $y = x^2$. $\therefore \frac{dy}{dx} = 2x$, and $\Delta x = \frac{2x}{100}$.

$$\therefore \quad \text{Percentage error in } y = \frac{100}{y} \cdot \frac{dy}{dx} \cdot \Delta x = \frac{100}{x^2} \cdot 2x \cdot \frac{2x}{100} = 4 \text{ per cent.}$$

(*b*) Volume, $y = x^3$. $\therefore \frac{dy}{dx} = 3x^2$.

$$\therefore \quad \text{Percentage error in } y = \frac{100}{x^3} \cdot 3x^2 \cdot \frac{2x}{100} = 6 \text{ per cent.}$$

(2) In measuring an angle known to be 27° an error of 18′, or 0·3°, was made. What is the consequent error in the values of (*a*) $\sin x$, (*b*) $\cos x$, (*c*) $\tan x$, (*d*) $\log \sin x$, (*e*) $\log \cos x$, (*f*) $\log \tan x$?

$$0{\cdot}3^\circ = 0{\cdot}3 \times \frac{\pi}{180} \text{ radians} = 0{\cdot}00524 \text{ radians} = \Delta x, \text{ and } x = 27^\circ.$$

Therefore

(*a*) $\Delta y = \frac{d \sin x}{dx} \cdot \Delta x = 0{\cdot}00524 \cos 27^\circ$

$= 0{\cdot}00524 \times 0{\cdot}891 = 0{\cdot}0047$ (p. 195).

(*b*) $\Delta y = \frac{d \cos x}{dx} \cdot \Delta x = -0{\cdot}00524 \sin 27^\circ$

$= -0{\cdot}00524 \times 0{\cdot}4540 = -0{\cdot}0024$ (p. 195).

(*c*) $\Delta y = \frac{d \tan x}{dx} \cdot \Delta x = 0{\cdot}00524 \sec^2 27^\circ$

$= 0{\cdot}00524 \times (1{\cdot}1223)^2 = 0{\cdot}0066$ (p. 195).

(*d*) $\Delta y = \frac{d \log_{10} \sin x}{dx} \cdot \Delta x = 0{\cdot}00524 \times 0{\cdot}4343 \cot 27^\circ$

$= 0{\cdot}00524 \times 0{\cdot}4343 \times 1{\cdot}9626 = 0{\cdot}0045$ (p. 196).

(*e*) $\Delta y = \frac{d \log_{10} \cos x}{dx} \cdot \Delta x = -0{\cdot}00524 \times 0{\cdot}4343 \tan 27^\circ$

$= -0{\cdot}00524 \times 0{\cdot}4343 \times 0{\cdot}5095 = -0{\cdot}0012$ (p. 196).

(*f*) $\Delta y = \frac{d \log_{10} \tan x}{dx} \cdot \Delta x = 0{\cdot}00524 \times \frac{0{\cdot}8686}{\sin 54^\circ}$

$= 0{\cdot}00524 \times \frac{0{\cdot}8686}{0{\cdot}809} = 0{\cdot}0056$ (p. 196).

(3) A pole AC, measured as 25 feet high, is found to throw a shadow CB 32 feet long. Thence is calculated the angle of elevation, θ, of the sun (fig. 97). If the true height of the pole is 25 feet 2 inches, what is the error in the calculated value of θ?

$$\Delta b = 2 \text{ inches} = \tfrac{1}{6} \text{ foot}.$$

$$\therefore \quad \Delta\theta = \frac{1}{6}\frac{d\theta}{db}.$$

But $\qquad b = a \tan\theta, \quad \therefore \quad \dfrac{db}{d\theta} = a \sec^2\theta = 32 \sec^2\theta$

$$\therefore \quad \frac{d\theta}{db} = \frac{1}{32}\cos^2\theta = \frac{1}{32}\frac{a^2}{c^2} = \frac{1}{32} \times \frac{32^2}{32^2+25^2}$$

$$= \frac{32}{1649},$$

$$\therefore \quad \Delta\theta = \frac{32}{6 \times 1649} \text{ radians} = 11'.$$

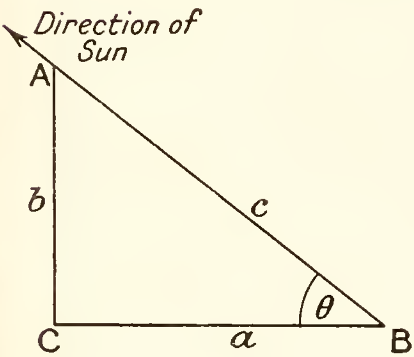


FIG. 97.

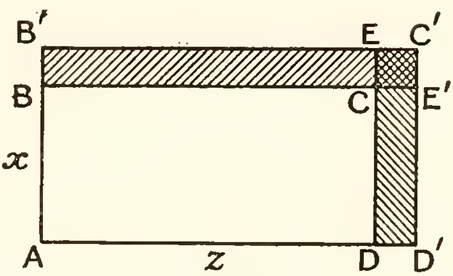


FIG. 98.

(4) The number of amperes of current (i) passing through a tangent galvanometer is given by $i = k \tan\theta$, where θ is the amount of deflection of the galvanometer needle and k is a constant. For which observed deflection of the needle will an error in reading of the galvanometer give the least error in the inferred value of the current?

$$\frac{di}{d\theta} = k \sec^2\theta,$$

$\therefore$ Percentage error in the inferred value of the current is

$$\frac{100\Delta i}{i} = \frac{100k \sec^2\theta \cdot \Delta\theta}{k \tan\theta} = \frac{100\Delta\theta}{\sin\theta\cos\theta} = \frac{200\Delta\theta}{\sin 2\theta}.$$

This will be a minimum when $\sin 2\theta$ is a maximum, *i.e.* when $2\theta = 90°$ or when $\theta = 45°$.

(2) **Cases of y = f(x, z, t, . . .).**—We will take as an illustration the function $y = xz$.

Let ABCD (fig. 98) be a rectangle whose sides are x and z.

Then its area $A = xz$.

Let AB′C′D′ be the rectangle whose sides are $(x+\Delta x)$ and $(z+\Delta z)$ where Δx and Δz are errors in the measurement of x and z.

Then, if we call the area AB′C′D′, $A+\Delta A$, we have

$$A+\Delta A=(x+\Delta x)(z+\Delta z)=xz+z\Delta x+x\Delta z+\Delta x\Delta z,$$

$$\therefore \quad \Delta A=z\Delta x+x\Delta z+\Delta x\Delta z$$

$$=\text{rectangles BB}'\text{EC}+\text{DD}'\text{E}'\text{C}+\text{EC}'\text{E}'\text{C}.$$

But $\Delta x\Delta z$, or the area EC′E′C, being very small compared with the other areas, may be neglected.

$$\therefore \quad \Delta A=z\Delta x+x\Delta z=\text{BB}'\text{EC}+\text{DD}'\text{E}'\text{C}.$$

But BB′EC is the error in the area due to the error in x (*i.e.* due to Δx) and DD′E′C is the error in the area due to the error in z (*i.e.* due to Δz).

∴ *The total error in area, viz. ΔA, is the sum of the errors resulting from the error in each of the sides (or variables) separately (assuming the other side, or variable, to be free from error).*

EXAMPLES.

(1) If in fig. 98 $x = 6{\cdot}57$ cm. and $z = 3{\cdot}53$ cm., each measurement being correct to two decimal places only, what is the largest error in the area of the rectangle?

$$x \text{ may be } 6{\cdot}57\pm0{\cdot}005, \text{ so that } \Delta x = \pm0{\cdot}005,$$

$$z \quad ,, \quad 3{\cdot}53\pm0{\cdot}005, \quad ,, \quad \Delta z = \pm0{\cdot}005.$$

$$\therefore \quad \Delta A = z\Delta x+x\Delta z = 3{\cdot}53\times\pm0{\cdot}005+6{\cdot}57\times\pm0{\cdot}005 = \pm0{\cdot}05 \text{ sq. cm.}$$

(2) The weight of a person has been found to be 20 kgrms., with a possible error of 0·25 kgrm., and his height 100 cm. with a possible error of 2 cm. What is the maximum percentage error in the estimated value of his surface area?

$$S = kW^{0\cdot425}\times H^{0\cdot725},$$

$$\therefore \quad \Delta S_1 \text{ (due to error in weight)} = \frac{dS}{dW}\cdot\Delta W = 0{\cdot}425kW^{-0\cdot575}\times H^{0\cdot725}\times0{\cdot}25$$

$$\text{and } \Delta S_2 \text{ (due to error in height)} = \frac{dS}{dH}\cdot\Delta H = 0{\cdot}725kW^{0\cdot425}\times H^{-0\cdot275}\times2,$$

∴ Total *percentage* error in S is

$$\frac{100(\Delta S_1+\Delta S_2)}{S} = \frac{100kW^{0\cdot425}\,.\,H^{0\cdot725}(0{\cdot}425\times0{\cdot}25W^{-1}+0{\cdot}725\times2H^{-1})}{kW^{0\cdot425}\,.\,H^{0\cdot725}}$$

$$=100\left(\frac{0{\cdot}425\times0{\cdot}25}{W}+\frac{0{\cdot}725\times2}{H}\right)$$

which, when $W = 20$ and $H = 100$, becomes

$$\frac{42{\cdot}5}{80}+1{\cdot}45 = 2 \text{ per cent.}$$

EXERCISES.

(1) In fig. 97 the height of the pole AC is obviously $a \tan \theta$. If $a = 100$ feet and θ is measured as $30°$, what is the error in the estimated height if the error in the angle be $1'$? How should one choose the distance a so as to make the percentage error in the estimated height as small as possible?

[*Answer*, $\Delta b = 0{\cdot}01745 a \sec^2 30° \,.\, \frac{1}{60}$ feet $= 1{\cdot}745 \times \frac{4}{3} \times \frac{1}{60}$ feet $= 0{\cdot}47$ inch. The percentage error $= \dfrac{1{\cdot}745 a \sec^2 \theta}{a \tan \theta} \,.\, \Delta\theta = \dfrac{3{\cdot}49 \Delta\theta}{\sin 2\theta}$, which is a minimum when $\theta = 45°$. $\therefore$ a should be chosen equal to b.]

(2) The daily number of calories (C) produced by a child is given by $1{\cdot}303 l \sqrt[3]{W^2}$ where l = length in cms. and W = weight in kgrms. If there is a possible error of 0·5 cm. in measuring the length of an infant as 52 cm. and one of 50 grms. in measuring its weight as 3·6 kgrms., what is the maximum error in the calculated value of the total heat production of that infant?

$$[\textit{Answer},\ \Delta C_1 + \Delta C_2 = 1{\cdot}303(3{\cdot}6^{\frac{2}{3}} \times 0{\cdot}5 + 52 \times \tfrac{2}{3} \times 3{\cdot}6^{-\frac{1}{3}} \times \tfrac{1}{20})$$

$$= \text{about 3 calories.}]$$

(3) An error of $0{\cdot}1\mu$ is made in measuring the diameter of a red blood corpuscle as $7{\cdot}4\mu$. What is the percentage error in its inferred area and volume respectively—assuming the corpuscle to be spherical?

[*Answer*, 2·7 per cent.; 4 per cent. approximately.]

CHAPTER XIII.

SUCCESSIVE DIFFERENTIATIONS.

In all the examples of differentiation that we considered in Chapter X. we saw that the differential coefficient also formed a function of x.

Thus, in the case of linear functions,

$$y = mx + b,$$

we get $$\frac{dy}{dx} = m, \quad i.e. \quad \frac{dy}{dx} = mx^0.$$

In the case of such functions as

$$y = x^2 + bx + c,$$

we get $$\frac{dy}{dx} = 2x + b.$$

When $$y = x^3 + ax^2 + bx + c,$$

we get $$\frac{dy}{dx} = 3x^2 + 2ax + b,$$

and so on.

Hence the differential coefficient of every function may itself be differentiated a number of times. Thus, let us take as an example

$$y = x^6 + 4x^3 + 3x + 4.$$

1st	differential coefficient	$= 6x^5 + 12x^2 + 3.$		
2nd	,, ,,	$= 6 \times 5x^4 + 12 \times 2x$	$= 30x^4 + 24x.$
3rd	,, ,,	$= 6 \times 5 \times 4x^3 + 12 \times 2 \times 1$	$= 120x^3 + 24.$
4th	,, ,,	$= 6 \times 5 \times 4 \times 3x^2$	$= 360x^2.$
5th	,, ,,	$= 6 \times 5 \times 4 \times 3 \times 2x$	$= 720x.$
6th	,, ,,	$= 720 \times 1$	$= 720.$
7th	,, ,,		$= 0.$

The notations employed for denoting the successive differential coefficients of the function $y = f(x)$ are as follows:—

1st differential coefficient is $y' = f'(x)$ or $\dfrac{dy}{dx}$.

2nd differential coefficient is $y'' = f''(x)$ or $\dfrac{d\left(\dfrac{dy}{dx}\right)}{dx}$, *i.e.*, $\dfrac{d^2y}{dx^2}$.

3rd ,, ,, is $y''' = f'''(x)$ or $\dfrac{d\left(\dfrac{d^2y}{dx^2}\right)}{dx}$, *i.e.*, $\dfrac{d^3y}{dx^3}$.

. . . .

nth ,, ,, $= f^n(x)$ or $\dfrac{d^ny}{dx^n}$.

From the above example it will be seen that for any rational algebraic function of the nth degree all the differential coefficients beyond the nth are zero. Thus in $y = x^6$, etc., the 7th and higher differential coefficients are zero. This is, however, not so in the case of irrational and transcendental functions. (Compare the various differential coefficients of e^x, sin x, cos x, etc., also of $\log_e x$.)

The Physical Meaning of $\dfrac{d^2y}{dx^2}$.—We have seen that $\dfrac{dy}{dx}$ represents the rate of change of a function, and if y represents space and x represents time, then $\dfrac{dy}{dx}$ represents *velocity* (*i.e.* rate of change of space with time). The second differential coefficient $\dfrac{d^2y}{dx^2}$ then represents rate of change of velocity, *i.e. acceleration*.

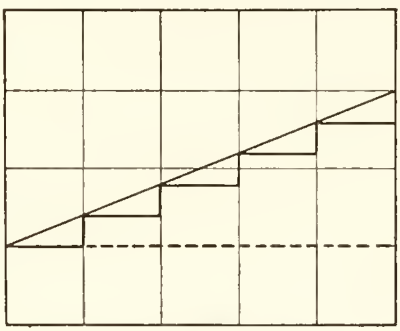


Fig. 99.—Constant Slope of a Straight Line.

If we represent a function by a curve, then $\dfrac{dy}{dx}$ means the slope of the curve and $\dfrac{d^2y}{dx^2}$ means the rate of change of slope.

Thus, in the case of a straight line, $y = mx + b$ (fig. 99), the slope $\dfrac{dy}{dx}$ is constant and $= m$,

$$\therefore \quad \frac{d^2y}{dx^2} = 0.$$

But in the case of a curve representing a function of higher

degree than the first, the slope, *i.e.* $\frac{dy}{dx}$, keeps on changing, *e.g.* in fig. 100, the slope is gradually increasing, whilst in fig. 101 it is gradually diminishing, and the rate at which the slope is changing is represented by $\frac{d^2y}{dx^2}$.

Concavity and Convexity.—Inspection of figs. 100 and 101

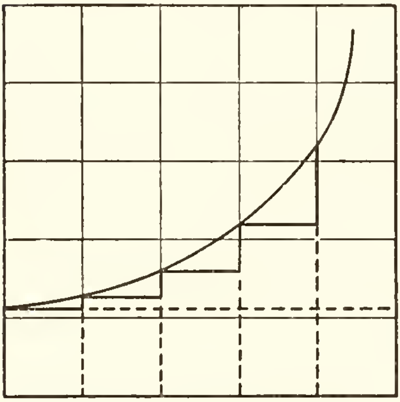


FIG. 100.—Increase of Slope in Case of Convex Curve.

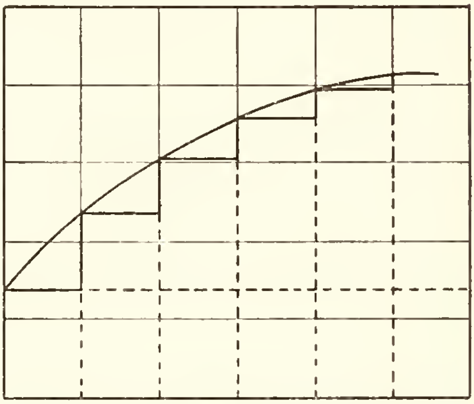


FIG. 101.—Decrease of Slope in Case of Concave Curve.

reveals an interesting fact, viz. that if $\frac{dy}{dx}$ gradually increases as one goes to the right, *i.e.* **when $\frac{d^2y}{dx^2}$ is positive, then the curve is convex downwards, and when $\frac{d^2y}{dx^2}$ is negative** $\left(\frac{dy}{dx}\right.$ diminishing to the right$\left.\right)$, **then the curve is convex upwards.**

Point of Inflection.—The point A or A′ (fig. 102), where the curve changes its shape from convexity to concavity, or *vice versa*, is called *a point of inflection*, and *at this point* $\frac{d^2y}{dx^2} = 0$.

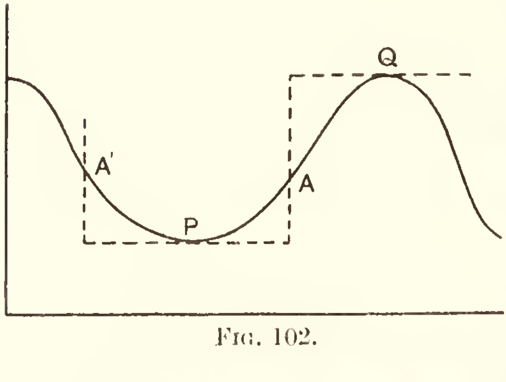


FIG. 102.

We may now summarise what we said on pp. 199–202 about maxima and minima and what we have just said about a point of inflection:

(i) At a *maximum or minimum* point on a curve, $\frac{dy}{dx} = 0$.

(ii) If from a point where $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2}$ *becomes positive* as we go to the right, then we know that that point was a *minimum*; whilst if $\frac{d^2y}{dx^2}$ *becomes negative* as we go to the right, then we know that the point was a *maximum*. This, as we have seen, is a matter of great importance in the consideration of problems on maxima and minima.

(iii) The point where $\frac{d^2y}{dx^2} = 0$ is a *point of inflection.*

EXAMPLES.

(1) Investigate the points of inflection of the curve $y = \sin x$.

Here $$\frac{dy}{dx} = \cos x,$$

and $$\frac{d^2y}{dx^2} = -\sin x.$$

$\therefore$ for a point of inflection we must have $-\sin x = 0$.

This condition is fulfilled at the points where

$$x = 0; \quad x = \pm\pi; \quad x = \pm 2\pi\,; \ldots x = \pm n\pi.$$

At these points $y = 0$.

$\therefore$ the points of inflection of this sine curve lie on the x axis at distances of π from one another. The points O, A, B, etc., are points of inflection (fig. 103).

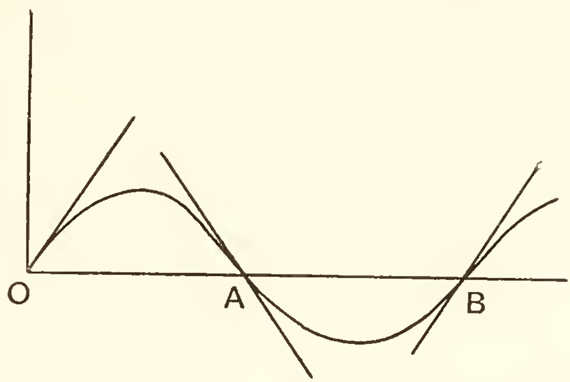


FIG. 103.

(2) Find the points of inflection of the normal curve of error whose equation is

$$y = Ae^{-\frac{x^2}{2\sigma^2}} \text{ (see p. 408).}$$

$$\frac{dy}{dx} = -\frac{Ax}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{A}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} + \frac{Ax}{\sigma^2}\cdot\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} = 0$$

$$\therefore \qquad \frac{A}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \left(\frac{x^2}{\sigma^2} - 1 \right) = 0,$$

whence $\qquad x = \pm \sigma.$

At these points $\qquad y = Ae^{-\frac{1}{2}}.$

Higher Differential Coefficients.

The differential coefficients higher than the first or second, viz. $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots \frac{d^ny}{dx^n}$, cannot be interpreted geometrically and have no physical meaning. They are, however, of great importance in mathematical analysis, because by means of sucessive differentiation we can expand any function in powers of x.

Maclaurin's or Stirling's Theorem.—We saw that the exponential series enables us to express the function $y = e^x$ in terms of a series of ascending powers of x (p. 80). Similarly we saw on p. 81 that the logarithmic series enables us to express the function $y = \log_e x$ in terms of a series of ascending powers of x. Indeed, it is possible to express *any* function of x in the form of a series

$$y = f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots,$$

and **Maclaurin's Theorem enables us—by means of successive differentiation—to find the values of the coefficients A, B, C, D, etc., in that series,** as follows:—

$$y = f(x) \quad = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \text{ (identically).}$$

$$\therefore \quad \frac{dy}{dx} = f'(x) \quad = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \text{ (identically).}$$

$$\therefore \quad \frac{d^2y}{dx^2} = f''(x) \quad = 2C + 3 \,.\, 2Dx + 4 \,.\, 3Ex^2 + \ldots \text{ (identically).}$$

$$\therefore \quad \frac{d^3y}{dx^3} = f'''(x) \quad = 3 \,.\, 2D + 4 \,.\, 3 \,.\, 2Ex + \ldots \text{ (identically).}$$

$$\therefore \quad \frac{d^4y}{dx^4} = f''''(x) = 4 \,.\, 3 \,.\, 2E + \ldots \text{ (identically).}$$

By making $x = 0$ in each of these identities, we obtain:

$$y = f(0) \quad = A,$$

$$\frac{dy}{dx} = f'(0) \quad = B,$$

$$\frac{d^2y}{dx^2} = f''(0) \quad = 2C, \quad \text{or} \quad \frac{1}{2\,!} f''(0) = C,$$

$$\frac{d^3y}{dx^3} = f'''(0) = 3 \,.\, 2\text{D}, \quad \text{or} \quad \frac{1}{3\,!}f'''(0) = \text{D},$$

$$\frac{d^4y}{dx^4} = f''''(0) = 4 \,.\, 3 \,.\, 2\text{E}, \quad \text{or} \quad \frac{1}{4\,!}f''''(0) = \text{E},$$

etc.

Maclaurin's theorem therefore states that, subject to certain limitations, any function of a single variable, x, in the form of $y = f(x)$, may be expressed in the form of the following series of ascending powers of x, viz.:

$$f(x) = f(0) + \frac{x}{1\,!}f'(0) + \frac{x^2}{2\,!}f''(0) + \frac{x^3}{3\,!}f'''(0) + \frac{x^4}{4\,!}f''''(0) + \ .\ .\ .$$

Note.—$f(0)$, $f'(0)$, $f''(0)$, etc. mean the values that $f(x)$, $f'(x)$, $f''(x)$, etc. assume when $x = 0$. Thus

(1) If $f(x) = e^x$, we have

$$\left.\begin{aligned} f(0) &= e^0 = 1 \\ \therefore\quad f'(0) &= e^0 = 1 \\ \therefore\quad f''(0) &= e^0 = 1 \\ \text{etc.}\quad & \text{etc.}\quad \text{etc.} \end{aligned}\right\} \text{since } \frac{d(e^x)}{dx} = e^x \text{ and } \frac{d^n(e^x)}{dx^n} = e^x. \text{ (See p. 184.)}$$

$$\therefore\quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ .\ .\ . = \text{Exponential Series.}$$

(2) If $f(x) = \sin x$, we have

$$\left.\begin{aligned} f(0) &= \sin 0^\circ &&= 0 \\ f'(0) &= \cos 0^\circ &&= 1 \\ f''(0) &= -\sin 0^\circ &&= 0 \\ f'''(0) &= -\cos 0^\circ &&= -1 \\ f''''(0) &= +\sin 0^\circ &&= 0 \\ \text{etc.} &\quad \text{etc.} && \text{etc.} \end{aligned}\right\} \begin{aligned} &\text{since } \frac{d \sin x}{dx} = \cos x \\ &\frac{d^2 \sin x}{dx^2} = \frac{d \cos x}{dx} = -\sin x \\ &\text{etc. (See p. 189.)} \end{aligned}$$

$$\therefore\quad \sin x = \frac{x}{1\,!} - \frac{x^3}{3\,!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ .\ .\ . \text{ to infinity } (x \text{ in radians}).$$

Similarly $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ .\ .\ .$ to infinity (x in radians), or, as an alternative method, $\cos x$ may be derived by differentiating $\sin x$, as follows:—

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ .\ .\ .$$

$$\therefore\quad \frac{d \sin x}{dx} \quad \text{or} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ .\ .\ .$$

Note.—Obviously cosec x cannot be expanded by Maclaurin's series, since cosec $0^\circ = \infty$.

It can also be shown that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3\,.\,5} + \ .\ .\ . \text{ and therefore } \mathrm{Lt}_{x \to 0} \frac{\tan x}{x} = 1,$$

$$\sin^{-1} x = x + \frac{x^3}{3!} + \frac{3x^5}{5!} + \ldots$$

and $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \text{ (Gregory's series).}$$

Putting $x = 1$, $\tan^{-1} x$ becomes $\pi/4$;

$$\therefore \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

Note.—Cot x, like cosec x, cannot be thus expanded, since $\cot 0° = \infty$.

To prove that $Lt_{x \to 0} \frac{\sin x}{x} = 1$, and $Lt_{x \to 0} \cos x = 1$.

Since $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\therefore \quad \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

$\therefore$ when $x \to 0$ the right-hand side of the identity becomes 1.
Similarly for $Lt_{x \to 0} \cos x$.

Calculation of the Values of sin x, cos x, etc.—Supposing we wish to find the value of sin 30°, then since 30° = 0·5236 radian, therefore $x = 0·5236$.

$\therefore$ putting this value of x in the series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

we get $$\sin 30° = (0·5236) - \frac{(0·5236)^3}{3!} + \frac{(0·5236)^5}{5!} -$$
$$= 0·5236 - 0·0239 + 0·0003 - \ldots = \mathbf{0·5000}.$$

Similarly $$\sin 60° = (1·0472) - \frac{(1·0472)^3}{3!} + \ldots \qquad = \mathbf{0·8660},$$

$$\cos 10° = 1 - \frac{(0·1745)^2}{2!} + \frac{(0·1745)^4}{4!} - \ldots = \mathbf{0·9848}.$$

and so on, for trigonometrical ratios of any other angle.

Exponential Values of sin x and cos x.—

Since $$e^{kx} = 1 + \frac{kx}{1} + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} + \frac{k^5x^5}{5!} + \ldots$$

$$\therefore \quad e^{ix} = 1 + \frac{ix}{1} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \ldots \quad (i = \sqrt{-1})$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \ldots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

(see p. 26),

i.e. $$e^{ix} = \cos x + i \sin x.$$

Similarly, $$e^{-ix} = \cos x - i \sin x,$$

$$\therefore \quad \frac{e^{ix}+e^{-ix}}{2} = \cos x \text{ (by addition),}$$

and $$\frac{e^{ix}-e^{-ix}}{2i} = \sin x \text{ (by subtraction).}$$

These results are important in the solution of certain types of differential equations (see p. 328).

EXAMPLES.

(1) If $y = (A+Bx)e^{-ax}$, prove that

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+a^2y = 0.$$

$$\frac{dy}{dx} = -ae^{-ax}(A+Bx)+Be^{-ax} \text{ (p. 173).}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \{a^2e^{-ax}(A+Bx) - Bae^{-ax}\} - Bae^{-ax}$$

$$= a^2e^{-ax}(A+Bx) - 2Bae^{-ax}$$

and $$2a\frac{dy}{dx} = -2a^2e^{-ax}(A+Bx)+2Bae^{-ax}$$

$$\begin{aligned}\therefore \quad \frac{d^2y}{dx^2}+2a\frac{dy}{dx}+a^2y &= \{a^2e^{-ax}(A+Bx) - 2Bae^{-ax}\} \\ &\quad + \{-2a^2e^{-ax}(A+Bx)+2Bae^{-ax}\} \\ &\quad + \{a^2e^{-ax}(A+Bx)\} \\ &= 0 \text{ identically.}\end{aligned}$$

This is a most important result, because it teaches us that if

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+a^2y = 0,$$

then $y = (A+Bx)e^{-ax}$, a fact of fundamental importance in the solution of differential equations of the second order (see p. 327). A and B are constants.

(2) Find the nth differential coefficient of $\log_e x$.

Since $$\frac{dy}{dx} = \frac{1}{x}, \quad \therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3y}{dx^3} = +\frac{2}{x^3},$$

$$\frac{d^4y}{dx^4} = -\frac{2 \,.\, 3}{x^4}, \quad \frac{d^5y}{dx^5} = +\frac{2 \,.\, 3 \,.\, 4}{x^5}, \quad \frac{d^6y}{dx^6} = -\frac{2 \,.\, 3 \,.\, 4 \,.\, 5}{x^6}.$$

. . . .

$$\therefore \quad \frac{d^ny}{dx^n} = (-1)^{n+1} \,.\, \frac{(n-1)!}{x^n}.$$

Leibnitz's Theorem.—Leibnitz's theorem is a theorem by means of which the *nth differential coefficient of a product of two functions of x* can be written down, provided we know the successive differential coefficients of the separate factors.

We have seen that if $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

$$\therefore \quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2} + \frac{dv}{dx}\cdot\frac{du}{dx},$$

$$= u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}.$$

$$\therefore \quad \frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + \frac{d^2v}{dx^2}\cdot\frac{du}{dx} + 2\frac{du}{dx}\cdot\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\cdot\frac{d^2u}{dx^2} + \frac{dv}{dx}\cdot\frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3}$$

$$= u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\cdot\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\cdot\frac{dv}{dx} + v\frac{d^3u}{dx^3}.$$

It will be noticed that the numerical coefficients in the above formulæ are the same as those occurring in the expansions of successive powers of $(a+b)^n$ by the binomial theorem.

Hence
$$\frac{d^ny}{dx^n} = \frac{d^n(uv)}{dx^n}$$

$$= u\frac{d^nv}{dx^n} + n\frac{du}{dx}\cdot\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{1\,.\,2}\,\frac{d^2u}{dx^2}\cdot\frac{d^{n-2}v}{dx^{n-2}}$$
$$+ \;.\;.\;.\; + n\frac{d^{n-1}u}{dx^{n-1}}\cdot\frac{dv}{dx} + v\frac{d^nu}{dx^n}.$$

This is *Leibnitz's* (or *Leibniz's*) *Theorem.*

The analogy between this theorem and Newton's binomial theorem is thus seen to be a very close one. It is remarkable that the two discoverers of the differential calculus should have discovered two such closely similar theorems.

Taylor's Theorem.—We have seen that Maclaurin's theorem, which expands certain functions of a *single* variable, x, into a series of ascending powers of that variable, fails when $f(0)$ is infinite, such as in the case of $\operatorname{cosec} x$, $\cot x$ and $\log x$ (since $\log 0 = -\infty$). In certain of these cases of failure of Maclaurin's theorem, the function can be expanded by means of *Taylor's theorem, which states that under certain conditions a function of the algebraic sum of* **two** *variables, such as $f(x+h)$, can be expanded into a series of ascending powers of one or other variable*, as follows:—

$$y = f(x \pm h) = f(x) \pm \frac{h}{1\,!}f'(x) + \frac{h^2}{2\,!}f''(x) \pm \frac{h^3}{3\,!}f'''(x) + \;.\;.\;.$$

Proof: Let $\quad y = f(x \pm h) \quad = A \pm Bh + Ch^2 \pm Dh^3 + \;.\;.\;.$. (i)

where A, B, C, etc., represent functions of x.

Differentiating successively with respect to h we get:

$$y' = f'(x \pm h) = \pm B + 2Ch \pm 3Dh^2 + \;.\;.\;.\;. \qquad \text{(ii)}$$
$$y'' = f''(x \pm h) = 2C \pm 3.2Dh + \;.\;.\;.\;. \qquad \text{(iii)}$$
$$y''' = f'''(x \pm h) = \pm 3.2D + \;.\;.\;.\;. \qquad \text{(iv)}$$

etc. etc. etc.

Putting $h = 0$ in each of these identities we obtain

From (i), $A = f(x)$; from (ii), $B = \pm\frac{f'(x)}{1\,!}$; from (iii), $C = \frac{f''(x)}{2\,!}$; from (iv), $D = \pm\frac{f'''(x)}{3\,!}$; etc.

$$\therefore \quad y = f(x \pm h) = f(x) \pm \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) \pm \frac{h^3}{3!}f'''(x) + \ldots$$

Note.—When $f(x)$, or any of its derivatives $f'(x)$, $f''(x)$, etc., becomes infinite, Taylor's theorem also fails.

EXAMPLES.

(1) $\log_e (x \pm h) = \log_e x \pm \frac{h}{1!}\left(\frac{1}{x} + \right)\frac{h^2}{2!}\left(-\frac{1}{x^2}\right) \pm \frac{h^3}{3!}\left(\frac{2}{x^3}\right) + \ldots$

$$= \log_e x \pm \frac{h}{x} - \frac{h^2}{2x^2} \pm \frac{h^3}{3x^3} - \ldots$$

When $x = 1$, $\log_e x = 0$, and we have

$\log_e (1 \pm h) = \pm\left(\frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} - \ldots\right)$, which is the logarithmic series (p. 81).

(2) Similarly, $\log_e \sin (x + h) = \log_e \sin x + h \cot x - \frac{1}{2}h^2 \operatorname{cosec}^2 x + \ldots$

Partial Differentiation.—We have so far dealt with the differentiation of functions of only *one* independent variable, and the differential coefficients we have obtained were the *complete* or *total* differential coefficients of those functions. In the study of biochemistry, however, one frequently has to deal with functions of more than one independent variable. Thus we know that the velocity of a chemical reaction depends not only upon the concentration of the reacting substances, but also upon the temperature; or that the volume of a gas depends not only upon the temperature, but also upon the pressure. Whenever we have to deal with such functions of more than one variable, we may differentiate the function with respect to one independent variable at a time, treating the other independent variables as if they were constants for the time being. Each of these differential coefficients is called a *partial* differential coefficient, and the total differential coefficient in such cases is a combination of the various partial coefficients.

Thus, in the case of a gas, we know that

PV = RT (where P = pressure, V = volume, T = absolute temperature, and R = constant).

$$\therefore \quad V = \frac{RT}{P},$$

$\therefore$ when T is constant

$$\frac{dV_T}{dP} = -\frac{RT}{P^2} \quad . \quad . \quad . \quad (1)$$

(the little T put as a subscript shows that T has been taken as a constant)

and when P is constant

$$\frac{dV_P}{dT} = \frac{R}{P} \quad . \quad . \quad . \quad . \quad (2)$$

A more convenient way of writing a partial differential coefficient is to use the Greek delta (δ) instead of d, when the subscript may be omitted.

Thus

$$\frac{\delta V}{\delta P} = -\frac{RT}{P^2} \quad . \quad . \quad . \quad . \quad (1a)$$

$\left(\frac{\delta V}{\delta P}\right.$ is called the coefficient of compressibility$\left.\right)$

and

$$\frac{\delta V}{\delta T} = \frac{R}{P} \quad . \quad . \quad . \quad . \quad (2a)$$

$\left(\frac{\delta V}{\delta T}\right.$ is called the coefficient of expansion.$\left.\right)$

Now, from (1) we have

$$dV_T = -\frac{RT}{P^2} \cdot dP$$

and from (1a)

$$-\frac{RT}{P^2} = \frac{\delta V}{\delta P}$$

$$\therefore \quad dV_T = \frac{\delta V}{\delta P} \cdot dP \quad . \quad . \quad . \quad (A)$$

Similarly,

$$dV_P = \frac{\delta V}{\delta T} \cdot dT \quad . \quad . \quad . \quad (B)$$

(A) and (B) are *partial differentials* of V.

The total variation of V, when both P and T vary together, is given by

$$\begin{aligned} dV &= dV_T + dV_P \\ &= \frac{\delta V}{\delta P} \cdot dP + \frac{\delta V}{\delta T} \cdot dT. \\ &= -\frac{RT}{P^2} \cdot dP + \frac{R}{P} \cdot dT. \end{aligned}$$

This is called the *total* or *complete differential* of V, and we thus see that

the total or complete differential of a function comprising two or more independent variables is equal to the sum of the partial differentials.

EXAMPLES.

(1) Find the partial differential coefficients of

$$z = \frac{x^3}{3} - 2x^3y - 2y^2x + \frac{y}{3},$$

and find the value of dz (the complete differential).

$$\frac{\delta z}{\delta x} = x^2 - 6yx^2 - 2y^2.$$

$$\frac{\delta z}{\delta y} = -2x^3 - 4yx + \tfrac{1}{3}.$$

$$dz = \frac{\delta z}{\delta x} \cdot dx + \frac{\delta z}{\delta y} dy$$

$$= (x^2 - 6yx^2 - 2y^2)dx - (2x^3 + 4yx - \tfrac{1}{3})dy.$$

(2) Find the total differential of $y = u^3 \sin v$.

$$\frac{\delta y}{\delta u} = 3u^2 \sin v; \quad \frac{\delta y}{\delta v} = u^3 \cos v.$$

$$\therefore \quad dy = 3u^2 \sin v\, du + u^3 \cos v\, dv.$$

(3) If $\qquad y = \sin\theta + \cos\phi$, then $\dfrac{\delta y}{\delta\theta} = \cos\theta; \quad \dfrac{\delta y}{\delta\phi} = -\sin\phi.$

$$\therefore \quad dy = \cos\theta\, d\theta - \sin\phi\, d\phi.$$

Maxima and Minima of Functions of More than One Independent Variable.—The conditions for maximum and minimum are that each partial differential coefficient should vanish (*i.e.* $=0$).

EXAMPLES.

(1) Find the maximum or minimum of

$$z = y + 2x - 2\log_e y - \log_e x.$$

$$\frac{\delta z}{\delta x} = 2 - \frac{1}{x}.$$

$$\frac{\delta z}{\delta y} = 1 - \frac{2}{y}.$$

For maximum or minimum

$$2 - \frac{1}{x} = 0, \text{ giving } x = \tfrac{1}{2},$$

and

$$1 - \frac{2}{y} = 0, \text{ giving } y = 2.$$

(2) Find maximum or minimum of

$$z = \frac{e^{x+y}}{xy}.$$

Since $e^{x+y} = e^x . e^y$

$$\therefore \quad \frac{\delta z}{\delta x} = \frac{1}{y} . \frac{xe^y e^x - e^x e^y}{x^2} = 0, \text{ giving } e^{x+y}(x-1) = 0, \text{ or } x = 1,$$

and $$\frac{\delta z}{\delta y} = \frac{1}{x} . \frac{ye^y e^x - e^x e^y}{y^2} = 0, \text{ giving } y - 1 = 0, \text{ or } y = 1.$$

(3) Find the relation which must subsist between the initial concentrations a and b when $(a+b)$ is constant, so that the velocity of reaction shall be a maximum in a bimolecular reaction.

$$\text{Velocity V} = \text{K}(a-x)(b-x).$$

$$\therefore \quad \frac{\delta \text{V}}{\delta a} = \text{K}(b-x) = 0, \text{ giving } b = x.$$

$$\frac{\delta \text{V}}{\delta b} = \text{K}(a-x) = 0, \text{ giving } a = x.$$

$\therefore$ condition is that $a = b$.

Note.—We have seen that if

$$z = f(x, y),$$

then

$$dz = \frac{\delta z}{\delta x}dx + \frac{\delta z}{\delta y}dy.$$

Hence if we are given an equation like

$$\text{M}dx + \text{N}dy = \text{A},$$

such an equation can only be a complete differential if

$$\text{M} = \frac{\delta z}{\delta x}, \quad \text{and} \quad \text{N} = \frac{\delta z}{\delta y}.$$

This is an important result in connection with the solutions of differential equations. (See Euler's Criterion, p. 319.)

CHAPTER XIV.

INTEGRAL CALCULUS.

The integral calculus may be considered as the inverse of the differential calculus. Thus, in the section on the differential calculus we concerned ourselves with the methods of differentiating any given function in order to ascertain the rate at which the dependent variable y changed with every momentary or infinitesimal change of the independent variable x in that function. The object of the integral calculus is the exact opposite of this, viz. to discover the original function from which the given differential coefficient or expression has been obtained.

Integration is the name given to this process of finding in terms of x the value of y from the given value of $\frac{dy}{dx}$, and is indicated by the symbol $\int$, which, being merely a long S, stands for "the sum of such quantities as." Thus, since dy stands for an infinitesimally small bit of y, therefore $\int dy$ (which is read "the integral of dy," and means the sum of all the infinite number of little bits "dy" of which y is made up), is equal to y, *i.e.* $\int dy = y$.

Similarly, $\int dx = x.$

Hence, given $\frac{dy}{dx} = 1,$

we have $dy = dx$ and $\int dy = \int dx$ or $y = x$. (But see p. 238, regarding addition of a constant.)

In many easy cases our knowledge of the differential calculus is sufficient to enable us to write down by mere inspection what is the original function whose differential coefficient is presented

to us. Thus, since when $y = x^2$, we get $\frac{dy}{dx} = 2x$, therefore we can say that when $\frac{dy}{dx} = 2x$, $y = x^2$, or if $dy = 2xdx$, then $y = x^2$.

But
$$\int dy = y.$$
$$\therefore \quad \int 2xdx = x^2.$$

Similarly, since when $y = x^4$, $\frac{dy}{dx} = 4x^3$,
$$\therefore \quad \int 4x^3dx = x^4.$$

Again, since when $y = \sin x$, $\frac{dy}{dx} = \cos x$,
$$\therefore \quad \int \cos xdx = \sin x.$$

If, therefore, we are familiar with the differential coefficient of any function, we can at once write down the original function of which the given expression is the differential coefficient.

The function to be integrated is called *the integrand.*

Addition of Constant.—There is, however, one point (to which we have already alluded on p. 237) of very great importance in connection with the integration of known differential expressions. Since an infinite number of functions which differ only in respect of the constant term have the same differential coefficient (see p. 172), it will be clear that if we work back from the differential coefficient to the original function, it will be necessary to add some symbolical constant, C, called the "**integration constant.**"

Thus
$$y = x^3$$
$$y = x^3 + \tfrac{1}{2},$$
$$y = x^3 - \tfrac{1}{3},$$
$$y = x^3 + 71,$$
and $y = x^3 + C$ (where C stands for any constant),

have all the same differential coefficient, viz. $\frac{dy}{dx} = 3x^2$, and therefore it is not quite correct to say that $\int 3x^2dx = x^3$, because

$x^3+\frac{1}{2}$, $x^3-\frac{1}{3}$, etc., would also have the same differential coefficient $\frac{dy}{dx} = 3x^2$, and therefore $\int 3x^2dx$ might also be $= x^3+\frac{1}{2}$ or $x^3-\frac{1}{3}$, or x^3+71, or x^3+C, etc., where C stands for any constant.

Hence we say that $\int 3x^2dx = x^3+C$.

Similarly, $\int 4x^3dx = x^4+C$,

$$\int \cos x dx = \sin x + C,$$

and so on.

Evaluation of the Integration Constant.—If no further data are given us, we cannot by mere inspection of the differential coefficient find out the value of C. In actual practice, however, some further information is given which makes it easy to evaluate this constant.

Thus, supposing we are given

$$\frac{dy}{dx} = 5x^4,$$

and we are told that when $x = 0$, $y = 7$, then we can at once write down the whole original function (including the constant term) which gave rise to $\frac{dy}{dx} = 5x^4$.

Thus $y = \int 5x^4dx = x^5+C$.

But when $x = 0$, $y = 7$ (by condition of the problem),

$\therefore$ when $x = 0$, $y = C = 7$.

$\therefore$ If $\frac{dy}{dx} = 5x^4$, and when $x = 0$, $y = 7$,

then $\int 5x^4dx = x^5+7$.

This is the same as saying that the equation of a curve which passes through the point (0, 7) and whose slope at any point is $5x^4$, is $y = x^5+7$.

In all the foregoing cases, if we indicate $\frac{dy}{dx}$ by y', then $dy = y'dx$, and $y = \int y'dx$.

The symbol $\int y'dx$ is read as "*the integral of y' with respect to x.*"

Geometrical Interpretation of C.—Consider the three straight lines of fig. 104.

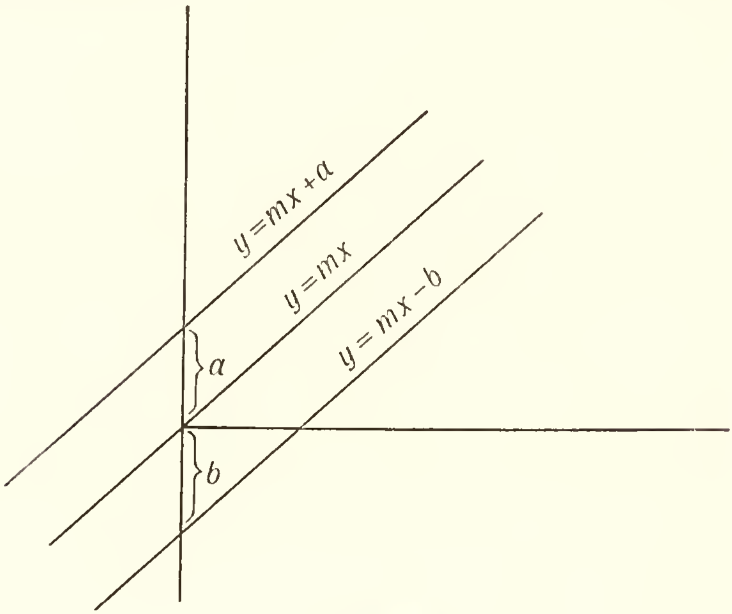


FIG. 104.

$$y = mx.$$
$$y = mx+a,$$
$$y = mx-b.$$

All of them are inclined to the x axis at the angle $\tan^{-1} m$.

$\therefore$ m is the differential coefficient (*i.e.* the slope) of all of them.

Hence, when we are asked to draw the graph whose differential coefficient is m, all we can do is to draw a line whose inclination to the x axis is $\tan^{-1} m$. Such a line might either pass through the origin ($y = mx$) or cut off an intercept a from the y axis ($y = mx+a$), or cut off any other intercept such as $-b$ from the same axis ($y = mx-b$). Hence, if we are told that $\frac{dy}{dx} = m$, we provisionally write down the equation of the original function as $y = mx+C$, where C may be any constant from $-\infty$ through 0 to $+\infty$. If, however, we are also told that the

graph cuts the y axis at a point (0, 3), then by writing down $y = mx + C$, and putting $x = 0$, we get

$$y = m \cdot 0 + C = C.$$

But when $x = 0,\ y = 3.$

$\therefore$ Original function is $y = mx + 3.$

Technique of Integration.

Algebraic Functions.

To find the integral of a power of x, such as x^m.

Since, when
$$y = \frac{x^{m+1}}{m+1}$$
$$\frac{dy}{dx} = (m+1) \cdot \frac{x^m}{m+1} \text{ (see p. 162)},$$
$$= x^m,$$
$$\therefore \quad dy = x^m \cdot dx,$$
$$\therefore \quad \int dy = \int x^m \cdot dx,$$

i.e.
$$y = \int x^m \cdot dx;$$

or
$$\frac{x^{m+1}}{m+1} = \int x^m \cdot dx.$$

Hence, the integral of $x^m = \dfrac{x^{m+1}}{m+1} + C.$

We therefore have the following *rule*:—

To find the integral of any power of x, add unity to the index (converting, for instance, x^m into x^{m+1}, divide the result by the index thus increased (i.e. by $m+1$) and add the integration constant C.

EXAMPLES.

$$\int x^2 dx = \frac{x^3}{3} + C.$$

$$\int x^7 dx = \frac{x^8}{8} + C.$$

$$\int x^{\frac{2}{3}} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{5}{3}}}{5/3} + C = \tfrac{3}{5}x^{\frac{5}{3}} + C.$$

$$\int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + C = -\frac{x^{-3}}{3} + C.$$

$$\int x^{-\frac{7}{8}}dx = \frac{x^{-\frac{7}{8}+1}}{-\frac{7}{8}+1}+\text{C} = \frac{x^{\frac{1}{8}}}{1/8}+\text{C} = 8x^{\frac{1}{8}}+\text{C}.$$

$$\int \tfrac{7}{9}x^{-\frac{2}{5}}dx = \tfrac{7}{9}\cdot\frac{x^{-\frac{2}{5}+1}}{-2/5+1}+\text{C} = \tfrac{7}{9}\cdot\frac{x^{\frac{3}{5}}}{3/5}+\text{C} = \tfrac{35}{27}x^{\frac{3}{5}}+\text{C}.$$

To verify any of these results differentiate them and one gets the expressions upon which the integration has been performed. Thus, to take three of the foregoing examples.

$$y = \tfrac{3}{5}x^{\frac{5}{3}}+\text{C}.$$
$$\therefore\quad \frac{dy}{dx} = \tfrac{5}{3}\times\tfrac{3}{5}x^{\frac{5}{3}-1} = x^{\frac{2}{3}}.$$
$$y = 8x^{\frac{1}{8}}+\text{C}.$$
$$\therefore\quad \frac{dy}{dx} = \tfrac{1}{8}\,.\,8x^{\frac{1}{8}-1} = x^{-\frac{7}{8}}.$$
$$y = \tfrac{35}{27}x^{\frac{3}{5}}+\text{C}.$$
$$\therefore\quad \frac{dy}{dx} = \tfrac{35}{27}\,.\,\tfrac{3}{5}x^{\frac{3}{5}-1} = \tfrac{7}{9}x^{-\frac{2}{5}}.$$

Exception to the foregoing Rule in the Case of x^{-1}.—Whilst the rule is generally true, there is one exception—and **one only**—to which this rule does not apply. This important exception is $\frac{1}{x}$ or x^{-1}. If we apply the ordinary rule we get

$$\int\frac{1}{x}dx = \int x^{-1}dx = \frac{x^{-1+1}}{-1+1}+\text{C}$$
$$= \frac{x^0}{0}+\text{C}$$
$$= \infty+\text{C}.$$

This is not an infinite but an *indefinite* expression, since the value of C may be anything between $-\infty$ and $+\infty$, and thus $\int\frac{1}{x}dx$ may be anything between 0 and ∞. But if we refer to p. 195, we find that $\frac{1}{x}$ is the differential coefficient of $\log_e x$.

Thus if $$y = \log_e x,$$
then $$\frac{dy}{dx} = \frac{1}{x};$$
$$\therefore\quad dy = \frac{1}{x}dx;$$
$$\therefore\quad \int dy = \int\frac{1}{x}dx,$$

i.e. $$y \text{ or } \log_e x = \int \frac{1}{x} dx.$$

$$\therefore \quad \int \frac{1}{x} dx = \log_e x + C.$$

This **logarithmic integral** is one of the most important ones in Biomathematics (see Chapter XV.).

There follows from it that $\int \frac{1}{a \pm x} dx = \pm \log_e (a \pm x) + C.$

Now take such an integral as $\int \frac{3x^2 dx}{1 + x^3}$.

Here we notice that $3x^2$ is the differential coefficient of $1 + x^3$, or the numerator, $3x^2dx$, is the differential of the denominator.

Hence by putting $1 + x^3 = z$ we obtain

$$\frac{dz}{dx} = 3x^2 \quad \text{or} \quad 3x^2dx = dz.$$

$$\begin{aligned} \therefore \quad \int \frac{3x^2dx}{1+x^3} &= \int \frac{dz}{z} \\ &= \log_e z + C \\ &= \log_e (1 + x^3) + C. \end{aligned}$$

Hence we obtain the following *most important rule*:—

If the numerator of a fraction is equal to the differential coefficient of the denominator, then the integral of the fraction is equal to the natural logarithm of the denominator + C.

Thus $$\int \frac{4x^3}{1+x^4} dx = \log_e (1 + x^4) + C$$

$$\int \frac{nx^{n-1}dx}{1+x^n} = \log_e (1 + x^n) + C.$$

(See further Examples on pp. 250 and 251.)

Exponential Functions.—

$\int e^x dx.$ Since when $$y = e^x$$

$$\frac{dy}{dx} = e^x,$$

$$\therefore \quad \int e^x dx = e^x + C.$$

$\int a^x dx.$ Let $$a = e^k.$$

$$\therefore \quad a^x = e^{kx}.$$

$$\therefore \quad \int a^x dx = \int e^{kx} dx.$$

Put $$kx = z.$$

Then $$\frac{dz}{dx} = k \quad \text{or} \quad dx = \frac{dz}{k}.$$

$$\therefore \quad \int e^{kx} dx = \int \frac{e^z dz}{k}$$

$$= \frac{1}{k}\int e^z dz$$

$$= \frac{1}{k}e^z + C.$$

But $$a = e^k,$$

$$\therefore \quad \log_e a = k;$$

$$\therefore \quad \frac{1}{k}e^z = \frac{e^z}{\log_e a}$$

$$= \frac{e^{kx}}{\log_e a}$$

$$= \frac{a^x}{\log_e a}.$$

$\therefore$ Finally $$\int a^x dx = \frac{a^x}{\log_e a} + C.$$

Trigonometrical Functions. (Circular and Inverse Circular.)

Since when $$y = \sin x$$

$$\frac{dy}{dx} = \cos x,$$

$$\therefore \quad \int \cos x dx = \sin x + C.$$

Also, when $$y = \cos x,$$

$$\frac{dy}{dx} = -\sin x.$$

$$\therefore \quad \int \sin x dx = -\cos x + C.$$

Similarly for all the other trigonometrical functions.

Fundamental Formulæ.—Collecting all the results so far obtained, we get the following table of formulæ:—

$y = \dfrac{x^{m+1}}{m+1}, \quad \dfrac{dy}{dx} = x^m$ (p. 241), $\quad \displaystyle\int x^m dx = \dfrac{x^{m+1}}{m+1} + C.$

except when $m = -1$.

$y = \log_e x, \quad \dfrac{dy}{dx} = \dfrac{1}{x}$ (p. 187), $\quad \displaystyle\int \frac{1}{x} dx = \log_e x + C.$

$y = \log_e (a \pm x), \quad \dfrac{dy}{dx} = \pm\dfrac{1}{(a \pm x)}, \quad \displaystyle\int \frac{dx}{a \pm x} = \pm \log_e (a \pm x) + C.$

$y = e^x, \quad \dfrac{dy}{dx} = e^x$ (p. 184), $\quad \displaystyle\int e^x dx = e^x + C.$

$y = \sin x, \quad \dfrac{dy}{dx} = \cos x$ (p. 189), $\quad \displaystyle\int \cos x dx = \sin x + C.$

$y = \pm \cos x, \quad \dfrac{dy}{dx} = \mp \sin x$ (p. 189), $\quad \displaystyle\int \sin x dx = -\cos x + C.$

$y = \tan x, \quad \dfrac{dy}{dx} = \sec^2 x$ (p. 189), $\quad \displaystyle\int \sec^2 x dx = \tan x + C.$

$y = \pm \cot x$, or $\cot(\pm x)$, $\quad \dfrac{dy}{dx} = \mp \operatorname{cosec}^2 x, \quad \displaystyle\int \operatorname{cosec}^2 x dx = -\cot x + C$, or $\cot(-x) + C.$

$y = \sin^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2 - x^2}}$ (p. 190), $\quad \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C;$

whence it follows that $\quad \displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C.$

$y = \tan^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{a}{a^2 + x^2}$ (p. 191), $\quad \displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C;$

whence it follows that $\quad \displaystyle\int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$

$y = \sec^{-1} \dfrac{x}{a}, \quad \dfrac{dy}{dx} = \dfrac{a}{x\sqrt{x^2 - a^2}}$ (p. 191), $\quad \displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C;$

whence it follows that $\quad \displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C.$

The following additional integrals can be verified by the reader by means of differentiation:—

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log_e \frac{x + \sqrt{x^2 + a^2}}{a} + C.$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e \frac{x + \sqrt{x^2 - a^2}}{a} + C.$$

$$\int\sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

$$\int\sqrt{a^2+x^2}dx = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log_e\frac{x+\sqrt{x^2+a^2}}{a}+C.$$

$$\int\sqrt{x^2-a^2}dx = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log_e\frac{x+\sqrt{x^2-a^2}}{a}+C.$$

Thus if $$y = \log_e\frac{x+\sqrt{x^2+a^2}}{a},$$

then $$\frac{dy}{dx} = \frac{a}{x+\sqrt{x^2+a^2}}\left(\frac{1}{a}+\frac{x}{a\sqrt{x^2+a^2}}\right) = \frac{1}{\sqrt{x^2+a^2}};$$

also if $$y = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log_e\frac{x+\sqrt{x^2-a^2}}{a},$$

then $$\frac{dy}{dx} = \frac{\sqrt{x^2-a^2}}{2}+\frac{x^2}{2\sqrt{x^2-a^2}}-\frac{a^2}{2}\frac{1}{\sqrt{x^2-a^2}},$$

$$= \sqrt{x^2-a^2}.$$

Integration of a Function containing a Constant as a Factor.—

If $$\frac{dy}{dx} = af(x), \text{ where } a \text{ is a constant,}$$

then $$\int af(x)dx = a\int f(x)dx.$$

In other words, **if there is a constant factor in the integrand then the constant can always be placed outside the integration sign.**

Thus $$\int 5x^4dx = 5\int x^4dx = 5\frac{x^5}{5}+C$$
$$= x^5+C.$$

Integration of a Constant by Itself.

Since when $$y = ax$$
$$\frac{dy}{dx} = a,$$
$$\therefore \int adx = ax.$$

Integration of the Algebraic Sum of Several Functions.—The algebraical sum of several functions is integrated by integrating

each function separately, adding them together (algebraically) and then adding the constant of integration.

Thus

$$\int\left(x^4 - 3x^3 + 7x^2 + 5x + \frac{a}{x} + \frac{b}{2\sqrt{x}} + 3\right)dx$$

$$= \int x^4dx - 3\int x^3dx + 7\int x^2dx + 5\int xdx + a\int\frac{1}{x}dx + \frac{b}{2}\int\frac{1}{\sqrt{x}}dx + 3\int dx,$$

$$= \frac{x^5}{5} - \frac{3}{4}x^4 + \frac{7}{3}x^3 + \frac{5}{2}x^2 + a\log_e x + \frac{b}{2}\cdot 2x^{\frac{1}{2}} + 3x + \mathrm{C},$$

$$= \frac{x^5}{5} - \frac{3}{4}x^4 + \frac{7}{3}x^3 + \frac{5}{2}x^2 + a\log_e x + b\sqrt{x} + 3x + \mathrm{C}.$$

Note.—Although when each of the separate functions $\int x^4dx$, $3\int x^3dx$, etc., is evaluated a constant must be added to each, thus: $\int x^4dx = \frac{x^5}{5} + \mathrm{C}_1$, $3\int x^3dx = \frac{3}{4}x^4 + \mathrm{C}_2$, etc., yet when integrating their algebraic sum it is sufficient to add only one constant C, since the algebraic sum of all the separate constants $\mathrm{C}_1 + \mathrm{C}_2 + \ldots$ is in itself a constant, and C can be considered to represent this sum.

Similarly,

$$\int\left(\frac{7x^6 + 3x^4 + 2x - 3}{x^2}\right)dx = \frac{7x^5}{5} + x^3 + 2\log_e x + \frac{3}{x} + \mathrm{C}.$$

EXAMPLES.

(1) Find the integral of $y' = \dfrac{x^4 + x^2 + 1}{1 + x^2}$.

$$\int\left(\frac{x^4 + x^2 + 1}{1 + x^2}\right)dx = \int\left[\frac{x^2(1 + x^2) + 1}{1 + x^2}\right]dx$$

$$= \int\left(x^2 + \frac{1}{1 + x^2}\right)dx$$

$$= \int x^2dx + \int\frac{1}{1 + x^2}dx$$

$= \frac{1}{3}x^3 + \tan^{-1} x + \mathrm{C}$ (see p. 245).

(2) Find the integral of $y' = 3x^4 - 4\cos x$.

$$\int(3x^4 - 4\cos x)dx = 3\int x^4dx - 4\int\cos x\,dx = \frac{3}{5}x^5 - 4\sin x + \mathrm{C}.$$

(3) Find the integral of $y' = \dfrac{3}{\sin^2 x - \sin^4 x}$.

$$\int \frac{3}{\sin^2 x - \sin^4 x}dx = 3\int \frac{1}{\sin^2 x(1 - \sin^2 x)}dx$$
$$= 3\int \frac{1}{\sin^2 x \cos^2 x}dx$$
$$= 3\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx$$
$$= 3\int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right)dx$$
$$= 3\int \frac{1}{\cos^2 x}dx + 3\int \frac{1}{\sin^2 x}dx$$
$$= 3\int \sec^2 x dx + 3\int \operatorname{cosec}^2 x dx$$
$$= 3 \tan x - 3 \cot x + C \text{ (see p. 245).}$$

(4) Integrate the following expression :—

$$y' = 4x^3 + 3 \tan^2 x.$$
$$\int (4x^3 + 3 \tan^2 x)dx = \int \left(4x^3 + 3\frac{\sin^2 x}{\cos^2 x}\right)dx$$
$$= 4\int x^3 dx + 3\int \frac{\sin^2 x}{\cos^2 x}dx$$
$$= x^4 + 3\int \frac{(1 - \cos^2 x)}{\cos^2 x}dx$$
$$= x^4 + 3\int (\sec^2 x - 1)dx$$
$$= x^4 + 3\int \sec^2 x dx - 3\int dx$$
$$= x^4 + 3 \tan x - 3x + C.$$

(5) Find the value of $\displaystyle\int \frac{1 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}dx$

$$\int \frac{1 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}dx = \int \left(\frac{1}{\sqrt{1 - x^2}} + 1\right)dx$$
$$= \sin^{-1} x + x + C \text{ (see p. 245).}$$

(6) Integrate $\sin 2x dx$.

Put $$2x = z.$$
$$\therefore \quad \frac{dz}{dx} = 2,$$
$$\therefore \quad dx = \frac{dz}{2}.$$

$$\therefore \quad \int \sin 2x dx = \int \frac{\sin z}{2} \cdot dz$$

$$= \tfrac{1}{2}\int \sin z\, dz$$

$$= -\tfrac{1}{2}\cos z + C$$

$$= -\tfrac{1}{2}\cos 2x + C.$$

Similarly, $$\int \sin ax dx = -\frac{1}{a}\cos ax + C.$$

(7) Integrate $$\int \frac{1}{(a+bx)^n} dx.$$

Put $$a + bx = z,$$

then $$dx = \frac{dz}{b}.$$

$$\therefore \quad \int \frac{1}{(a+bx)^n} dx = \int \frac{1}{bz^n} \cdot dz$$

$$= \frac{1}{b}\int z^{-n} dz$$

$$= \frac{1}{b(1-n)} \cdot z^{(1-n)} + C$$

$$= \frac{1}{b(1-n)}(a+bx)^{1-n} + C$$

$$= \frac{1}{b(1-n)} \cdot \frac{1}{(a+bx)^{n-1}} + C.$$

(8) Find the value of $\int e^{ax} dx$.

Put $$ax = z,$$

$$\therefore \quad dx = \frac{dz}{a}.$$

$$\therefore \quad \int e^{ax} dx = \frac{1}{a}\int e^z dz$$

$$= \frac{1}{a} e^z + C$$

$$= \frac{1}{a} e^{ax} + C.$$

(9) Integrate $$\frac{x^3 dx}{x^2 - 3x + 2}.$$

Divide the numerator by the denominator until the numerator contains a lower power of x than the denominator.

Thus
$$\frac{x^3}{x^2-3x+2} = x+3+\frac{7x-6}{x^2-3x+2},$$

$$\therefore \quad \int\frac{x^3}{x^2-3x+2}dx = \int(x+3)dx + \int\frac{7x-6}{x^2-3x+2}dx$$
$$= \frac{x^2}{2}+3x+\int\frac{7x-6}{x^2-3x+2}dx.$$

By resolving $\dfrac{7x-6}{x^2-3x+2}$ into partial fractions (p. 30),

we get
$$\frac{7x-6}{x^2-3x+2} = \frac{8}{x-2}-\frac{1}{x-1}.$$

$$\therefore \quad \int\frac{7x-6}{x^2-3x+2}dx = \int\frac{8}{x-2}dx - \int\frac{1}{x-1}dx,$$
$$= 8\log_e(x-2)-\log_e(x-1).$$

$$\therefore \quad \int\frac{x^3}{x^2-3x+2}dx = \frac{x^2}{2}+3x+8\log_e(x-2)-\log_e(x-1)+C.$$

(10) Integrate $\dfrac{5x^6}{1+x^7}dx.$

$$\int\frac{5x^6}{1+x^7}dx = \int\frac{\frac{5}{7}\,.\,7x^6}{1+x^7}dx$$
$$= \tfrac{5}{7}\int\frac{7x^6}{1+x^7}dx$$
$$= \tfrac{5}{7}\log_e(1+x^7)+C \text{ (see p. 243).}$$

(11)
$$\int\frac{xdx}{1+x^2} = \tfrac{1}{2}\log_e(1+x^2)+C \text{ (see p. 243)}$$
$$= \log_e\sqrt{1+x^2}+C.$$

(12) Find the value of $\int \tan x dx$ and $\int \cot x dx$.

$$\int \tan x dx = \int\frac{\sin x}{\cos x}dx.$$

But
$$\sin x = -\frac{d\cos x}{dx}.$$

$$\therefore \quad \int\tan x dx = -\int\frac{d\cos x}{\cos x} = -\log_e\cos x + C.$$

$$\int\cot x dx = \int\frac{\cos x}{\sin x}dx.$$

But
$$\cos x = \frac{d\sin x}{dx},$$

$$\therefore \quad \int\cot x dx = \log_e\sin x + C.$$

The following is an instructive example of how a fraction can be manipulated so as to make the numerator the differential coefficient of the denominator and thus render it suitable for integration (see p. 243).

(13) Find the value of $\displaystyle\int\frac{d\theta}{\sin(\beta-\theta)}$.

$$\frac{1}{\sin(\beta-\theta)} = \frac{1}{2\sin\frac{1}{2}(\beta-\theta)\,.\cos\frac{1}{2}(\beta-\theta)}$$

$$= \frac{\sin\frac{1}{2}(\beta-\theta)}{2\sin^2\frac{1}{2}(\beta-\theta)\,.\cos\frac{1}{2}(\beta-\theta)}$$

$$= \tfrac{1}{2}\operatorname{cosec}^2\tfrac{1}{2}(\beta-\theta)\,.\tan\tfrac{1}{2}(\beta-\theta)$$

$$= \frac{\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta-\theta)}{\cot\frac{1}{2}(\beta-\theta)}.$$

But $\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta-\theta)$ is the differential coefficient of $\cot\frac{1}{2}(\beta-\theta)$,

$$\therefore\quad \int\frac{d\theta}{\sin(\beta-\theta)} \text{ which } = \int\frac{\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta-\theta)d\theta}{\cot\frac{1}{2}(\beta-\theta)}$$

$$= \log_e\cot\tfrac{1}{2}(\beta-\theta)+\mathrm{C}.$$

This is a very important integral (see p. 290).

(14) The pressure on the surface of a lake due to the atmosphere is known to be 14 lbs. per square inch. The pressure, p, in the liquid x inches below the surface is known to be given by the law $dp/dx = 0{\cdot}036$. Find the pressure in the liquid at a depth of 10 feet.

$$\int dp = 0{\cdot}036\int dx+\mathrm{C}$$

$$\therefore\quad p = 0{\cdot}036x+\mathrm{C}.$$

But at the surface, where $x = 0$, $p = 14$; $\therefore$ C = 14.
$\therefore$ $p = 0{\cdot}036x+14$.
$\therefore$ When $x = 10$ feet, or 120 inches, $p = 0{\cdot}036\times120+14 = 18{\cdot}3$ lbs. per square inch.

(15) A flexible beam of wood ABC (fig. 105) 20 inches long has a part

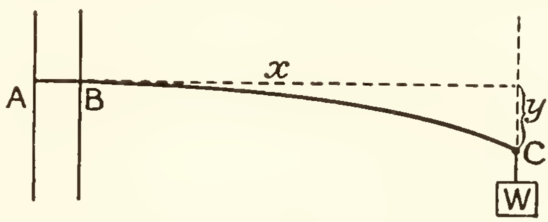


Fig. 105.

AB 2 inches long imbedded in the wall, so that AB is horizontal. A weight W is hung at C and it is known that the deflection y inches from the horizontal, at a distance x inches from B, obeys the law $d^2y/dx^2 = k(18-x)$.

(i) Find dy/dx at any point x.
(ii) Find y at any point x.
(iii) Find k, being given that the deflection of C is 3 inches.
(iv) Find the deflection half-way along the beam.
(v) Find the gradient half-way along the beam.

(i) Since $$\frac{d^2y}{dx^2} = k(18 - x),$$

$$\therefore \quad \frac{dy}{dx} = k\left(18x - \frac{x^2}{2}\right).$$

(ii) $$\therefore \quad y = k\left(9x^2 - \frac{x^3}{6}\right).$$

(iii) Since, when $x = 18$ (*i.e.* when the weight is at C), $y = 3$,

$$\therefore \quad 3 = k\left(9 \,.\, 18^2 - \frac{18^3}{6}\right) \text{ whence } k = \frac{1}{648}.$$

$$\therefore \quad \text{Law connecting } y \text{ and } x \text{ is } y = \frac{1}{648}\left(9x^2 - \frac{x^3}{6}\right).$$

(iv) $\therefore$ The deflection (*i.e.* value of y) half-way along the beam, *i.e.* when $x = 9$, is

$$y = \frac{1}{648}\left(9 \times 9^2 - \frac{9^3}{6}\right) = \frac{15}{16} \text{ inch.}$$

(v) The gradient $\dfrac{dy}{dx} = \dfrac{1}{648}\left(18x - \dfrac{x^2}{2}\right)$.

$$\therefore \quad \text{Half-way along beam, when } x = 9,\ \frac{dy}{dx} = \frac{1}{648}\left(18 \times 9 - \frac{9^2}{2}\right) = \frac{3}{16}.$$

$$\therefore \quad \text{Slope} = \tan^{-1} \tfrac{3}{16} = \tan^{-1} 0{\cdot}1875 = 10^\circ\, 37'.$$

The slope at the extremity is given by $\dfrac{dy}{dx} = \dfrac{1}{648}\left(18 \times 18 - \dfrac{18^2}{2}\right) = 0{\cdot}25$;

$$\therefore \quad \text{Slope} = \tan^{-1} 0{\cdot}25 = 14^\circ\, 2'.$$

Definite Integral.—All the integrals we have considered up to now, viz. those of the form $\int f(x) \,.\, dx$, are called *indefinite*, or *general*, integrals, because when the integration has been performed, an expression is obtained which is another function of x of the form F(x), and whose value is undetermined so long as the value of x is undetermined. If, however, we write the integral in the form $\int_b^a f(x)dx$, we get a *definite integral*, because the symbol $\int_b^a$ tells us that, having found the expression F(x), of which the function $f(x)$ is the differential coefficient, we are first to substitute a for x, then substitute b for x, and finally subtract the latter from the former value.

A definite integral of the form $\int_b^a f(x)dx$ is read as follows: "The integral of $f(x)dx$ between the limits a and b." The upper value (a) is called the *superior limit*, and the lower value (b) is called the *inferior limit*.

A few examples will make this clear.

EXAMPLES.

(1) Find the value of $\int_2^4 x^3dx$.

The general or indefinite integral is, of course, $\frac{1}{4}x^4 + C$.
Putting $x = 4$ we get $\frac{1}{4} . 4^4 + C = \frac{1}{4} . 256 + C = 64 + C$.
Putting $x = 2$ we get $\frac{1}{4} . 2^4 + C = \frac{1}{4} . 16 + C = 4 + C$.
Subtracting $4 + C$ from $64 + C$ we get 60.

$$\therefore \quad \int_2^4 x^3dx = 60.$$

Generally $\quad \int_b^a f(x)dx = F(a) - F(b).$

Similarly $\quad \int_b^a f(y)dy = F(a) - F(b).$

$$\therefore \quad \int_b^a f(x)dx = \int_b^a f(y)dy = \int_b^a f(z)dz, \text{ etc.}$$

It will be noticed that the **integration constant,** which is always added in the general integral, **disappears by subtraction in the definite integral.**

Note.—Since $\quad \int_b^a f(x)dx = F(a) - F(b),$

and $\quad \int_a^b f(x)dx = F(b) - F(a),$

$$\therefore \quad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

(2) Find the work done by a gas in expanding under constant temperature from 3 to 4 cubic feet, assuming it to obey Boyle's law (PV = constant) and being given that when V = 0·3 cubic foot P = 2000 lbs. per square foot.

$$PV = 0{\cdot}3 \times 2000 = 600, \text{ whence } P = \frac{600}{V}$$

Work, in foot-pounds, done by gas in expanding through volume $dV = PdV$.
$\therefore$ Work, in foot-pounds, done by gas in expanding through volume V

$$= \int PdV$$

$$= 600\int\frac{dV}{V}$$

$\therefore$ Work, in foot-pounds, done by gas in expanding

$$\text{from 3 to 4 cubic feet} = 600\int_4^3 \frac{dV}{V}$$
$$= -600(\log_e 4 - \log_e 3)$$
$$= -600 \times 2{\cdot}3(\log_{10} 4 - \log_{10} 3)$$
$$= -172{\cdot}5 \text{ ft.-lbs.}$$

Note.—The minus sign indicates that the work is done *by* the gas in expanding, this being *numerically* equal to the work done *on* the gas in compressing it through the same volume interval.

(3) Evaluate the following important integrals (see p. 311):—

$$\text{(i)} \int_0^\infty \frac{dx}{2(1+x^2)}, \qquad \text{(ii)} \int_0^\infty xe^{-x^2(1+v^2)}dx$$

$$\text{(i)} \int_0^\infty \frac{dx}{2(1+x^2)} = \left[\tfrac{1}{2}\tan^{-1}x\right]_0^\infty \text{(see p. 245)} = \tfrac{1}{2}\tan^{-1}\infty - \tfrac{1}{2}\tan^{-1}0 = \frac{\pi}{4}$$

$$\text{(ii) Since } \frac{d\,\dfrac{e^{-x^2(1+v^2)}}{2(1+v^2)}}{dx} = -\frac{2x(1+v^2)e^{-x^2(1+v^2)}}{2(1+v^2)} = -xe^{-x^2(1+v^2)}$$

$$\therefore \int_0^\infty xe^{-x^2(1+v^2)}dx = -\left[\frac{e^{-x^2(1+v^2)}}{2(1+v^2)}\right]_0^\infty = \frac{1}{2(1+v^2)}$$

Poisseuille's Law for the Flow of a Viscous Liquid through a Tube.

If v = velocity of a layer of the liquid parallel to the axis and at a distance r from it, then the tangential stress due to viscosity (per unit area) $= \eta\frac{dv}{dr}$ (η = coefficient of viscosity).

As total area of a tube of length l and radius $r = 2\pi rl$ (p. 57),

$\therefore$ Total retardation over length l is $F = -2\pi rl\eta\frac{dv}{dr}$ (retardation being opposed to the velocity must have a minus sign).

But F = difference of thrusts at ends of the tube $= \pi r^2 p$ (where p = pressure of fluid inside the tube).

$$\therefore \quad \pi r^2 p = -2\pi rl\eta\frac{dv}{dr},$$

or
$$-l\eta dv = \tfrac{1}{2}prdr,$$

i.e.
$$-l\eta\int dv = \tfrac{1}{2}p\int rdr,$$

or
$$-l\eta v = \tfrac{1}{4}pr^2 + C.$$

But when the layer is at the periphery, $r = \mathrm{R}$ (radius of tube) and velocity $v = 0$.

$$\therefore \quad \mathrm{C} = -\tfrac{1}{4}p\mathrm{R}^2.$$

$$\therefore \quad l\eta v = \tfrac{1}{4}p(\mathrm{R}^2 - r^2).$$

$$\therefore \quad v = \frac{p(\mathrm{R}^2 - r^2)}{4l\eta}.$$

Now, cross-section of annulus of thickness dr is $2\pi r dr$.

$\therefore$ Volume of flow (per unit of time) due to this annulus is given by the equation:

$$d\mathrm{V} = \frac{p(\mathrm{R}^2 - r^2)2\pi r dr}{4l\eta}$$

$$= \frac{p\pi r(\mathrm{R}^2 - r^2)dr}{2l\eta}.$$

$\therefore$ Total volume of flow from the tube per unit of time is

$$\mathrm{V} = \int_0^{\mathrm{R}} \frac{p\pi r(\mathrm{R}^2 - r^2)dr}{2l\eta}$$

$$= \frac{p\pi}{2l\eta}\int_0^{\mathrm{R}} r(\mathrm{R}^2 - r^2)dr$$

$$= \frac{p\pi}{2l\eta}\left[\mathrm{R}^2\int_0^{\mathrm{R}} rdr - \int_0^{\mathrm{R}} r^3 dr\right]$$

$$= \frac{p\pi}{2l\eta}\left(\frac{\mathrm{R}^4}{2} - \frac{\mathrm{R}^4}{4}\right)$$

$$= \frac{p\pi\mathrm{R}^4}{8l\eta},$$

$$\therefore \quad p = \frac{8l\eta\mathrm{V}}{\pi\mathrm{R}^4} \quad \text{or} \quad \eta = \frac{\pi p\mathrm{R}^4}{8\mathrm{V}l},$$

which is Poisseuille's law.

CHAPTER XV.

BIOCHEMICAL APPLICATIONS OF INTEGRATION.

Value of Integration.—Integration serves many useful and valuable purposes. In scientific work one frequently forms an hypothesis regarding the process of a certain phenomenon. Such an hypothesis is expressed in the form of a differential equation. In order to test the validity of the hypothesis, however, the differential equation is in itself of no use, because a differential coefficient expresses an **instantaneous** rate of change which, of course, it is impossible to measure experimentally. If, however, by means of integration, we can convert the differential expression into some relation between y and x in which no differentials are present, we can at once subject this relationship to the test of experiment and compare the observed with the calculated results.

As an example let us consider the case of two interacting chemical substances A and B, giving rise to the substances A′ and B′. Such transformation does not occur instantaneously.

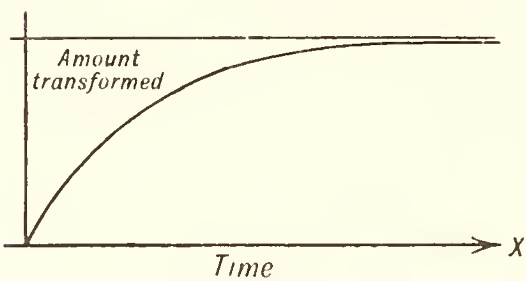


FIG. 106.

If we plot a graph (fig. 106) showing the amount of transformation in time t we get a curve like the one in the diagram. From this graph we learn the following facts, viz.:

(i) At the beginning of the reaction, when $t = 0$, the amount of substance transformed $= 0$.

(ii) The reaction starts rapidly and then gradually slows off. This is shown by the fact that the curve, which is steep at first, gradually becomes flatter and flatter.

Now imagine the reaction to occur between the molecules of A and B in such a way that single molecules of A and B are respectively transformed into single molecules of A′ and B′. Then,

since reaction occurs between molecules in contact, it is clear that the rapidity of the reaction or transformation will depend upon the frequency with which A and B meet. In other words, **the reaction velocity, or the rate of change of concentration, must be proportional to the concentration of each of the reacting substances.** But when one quantity varies as, or is proportional to, two or more other quantities, then it varies as the product of these other quantities.

$$\therefore \quad \textbf{Reaction velocity} = \mathbf{KC_AC_B},$$

where C_A = concentration of A (*i.e.* number of gramme-molecules per litre),

C_B = ,, ,, B,

and K = constant.

This is *Guldberg and Waage's law of mass action.*

If the reaction takes place between two molecules of A and one molecule of B, according to the equation

$$2A + B = A' + 2B',$$

then, since the left-hand side of the equation may be written as A + A + B,

$$\therefore \quad \text{reaction velocity} = KC_AC_AC_B = KC_A{}^2C_B.$$

Similarly, if the equation for the reaction is

$$mA + nB + kC + lD + \ldots = m'A' + n'B' + k'C' + l'D' + \ldots,$$

the reaction velocity $= KC_A{}^mC_B{}^nC_C{}^kC_D{}^l \ldots$

Unimolecular Reactions.—Supposing now we have a reaction like the inversion of cane sugar,

$$C_{12}H_{22}O_{11} + H_2O = \underset{\text{Dextrose}}{C_6H_{12}O_6} + \underset{\text{Lævulose}}{C_6H_{12}O_6}$$

The reaction velocity $= KC_sC_w$,

where C_s = concentration of cane sugar,

C_w = concentration of water.

If we start with a dilute solution, then the change in the concentration of the water is negligible (see p. 92), and therefore KC_w remains constant during the reaction.

Let $KC_w = k$, $\therefore$ reaction velocity $= kC_s$.

But reaction velocity $= \dfrac{dx}{dt}$,

where x is the amount of cane sugar transformed during a time t. Hence, if a = the original amount of cane sugar, we have C_s, *i.e.* the concentration of cane sugar at any instant $= (a - x)$.

$$\therefore \quad \mathbf{\frac{dx}{dt} = k(a - x).}$$

This, then, is the theoretical expression for the reaction between cane sugar and water. Now it is clear that such an expression, or **differential equation,** does not lend itself to experimental verification, since the interval of time would have to be taken fairly large, say a good many minutes, to give any appreciable change of concentration, and during that time x would have gone on increasing and $a - x$ would have been correspondingly diminishing. But if we can integrate this expression and get a relation between x and t without any differentials, then it will be easy to subject the equation to the test of experiment. Let us therefore proceed to integrate it.

Since $$\frac{dx}{dt} = k(a - x),$$

$\therefore$ by **separating the variables,** *i.e.* by grouping all the x's on one side and all the t's on the other side, we get

$$\frac{dx}{a - x} = kdt.$$

$$\therefore \quad \int kdt = \int \frac{dx}{a - x},$$

i.e. $$kt = -\log_e (a - x) + \mathrm{C} \text{ (see p. 243).}$$

To find the value of the constant C put $t = 0$ (*i.e.* before the reaction has started), then x also $= 0$ (*i.e.* no transformation has taken place). We then have

$$k \cdot 0, \text{ i.e. } 0 = -\log_e (a - 0) + \mathrm{C},$$
$$\therefore \quad \mathrm{C} = \log_e a.$$

Hence we have finally

$$kt = \log_e a - \log_e (a - x) = \log_e \frac{a}{a - x},$$

whence $$k = \frac{2{\cdot}3}{t} \log_{10} \frac{a}{a - x} \text{ (see pp. 79 and 85),}$$

and since $\frac{k}{2{\cdot}3}$ is also a constant, we may, if we wish, write the reaction constant $k = \frac{1}{t} \log_{10} \frac{a}{a - x}$.

Hence, by measuring at different intervals $t_1, t_2, t_3, \ldots t_n$, the corresponding values of $x_1, x_2, x_3, \ldots x_n$, or of $(a-x_1)$, $(a-x_2)$, etc., we ought to find (if the original supposition of the law of mass action for a unimolecular reaction holds good,

viz. $\frac{dx}{dt} = k(a-x)$), that

$$\frac{1}{t_1}\log\frac{a}{a-x_1} = \frac{1}{t_2}\log\frac{a}{a-x_2} = \frac{1}{t_3}\log\frac{a}{a-x_3} = \ldots = k.$$

The following table shows the value of k at different times t (a is taken as 100, and 1435 minutes are taken as the unit of time). From this it will be seen that within the limits of experimental error k is constant, and therefore there is *prima facie* evidence that the law of mass action for a unimolecular reaction holds good in this particular case.

t (in minutes from start).	t (in units of time).	$(a-x)$.	$\frac{1}{t}\log_{10}\frac{a}{a-x} = k$.
1435	1	58·240	0·2348
4315	3·007	19·530	0·2359
7070	4·927	7·008	0·2343
11360	7·916	1·484	0·2310
14170	9·875	0·534	0·2301
16935	11·801	0·185	0·2316
19815	13·808	0·069	0·2291
29925	20·854	0·001	0·2398
Mean k			0·2333

EXAMPLES.

(1) Madsen and Famulener investigated the loss of activity of vibriolysin at 28° C. by measuring the hæmolytic power of a vibriolysin solution kept in a thermostat for different periods of time, $t_1, t_2, t_3 \ldots$

Now assuming that the loss of activity takes place in accordance with Guldberg and Waage's law for a monomolecular reaction, viz. that

$$k = \frac{1}{t}\log\frac{a}{a-x},$$

where a = original amount of lysine and x = amount of lysine destroyed in time t, one can easily calculate k, and by a transformation of the equation, we have $a - x = ae^{-kt}$ (see p. 85).

The following are the observed and calculated results of $a-x$ at different times :—

Time in minutes.	$(a-x)$ observed.	$(a-x)$ calculated.
0	100	100
10	78·3	83·2
20	67·6	69·5
30	59·3	57·9
40	49·8	48·3
50	40·8	40·8
60	34·4	33·6

from which it is seen that the agreement is very close, suggesting that the hypothesis is correct.

(2) After what interval of time will the initial concentration of a substance, undergoing chemical transformation in accordance with the law of a unimolecular reaction, be halved?

The equation for a unimolecular reaction is

$$kt = 2{\cdot}3 \log \frac{a}{a-x}.$$

When $x = \frac{1}{2}a$, we have

$$kt = 2{\cdot}3 \log \frac{a}{a/2} = 2{\cdot}3 \log 2.$$

$$\therefore \quad t = \frac{2{\cdot}3}{k} \log 2 = \frac{0{\cdot}69}{k}.$$

Hence we have the following *corollary* to the equation for a unimolecular reaction—**the time at which the reaction is half-complete, or p per cent. complete, is independent of the initial concentration in all cases of unimolecular reactions. It depends only upon the reaction constant k, with which it varies inversely.** This is a fact of great practical value when it is necessary to determine the order of any particular reaction.

For example, when diphtheria antitoxin was injected into the bodies of animals in various strengths and amounts, Bomstein found that the quantity of antitoxin present in the blood four days after injection was in every case the same fraction of the original amount. Hence, he inferred that the rate of disappearance of the antitoxin was according to the law of a unimolecular reaction. He confirmed this by plotting $\log n$ (n = content of antitoxin) against time, when a straight line was obtained (see Chapter XXII., p. 359 *et seq.*).

Bimolecular Reaction.—Let us now consider such a reaction as the following:—

$$\underbrace{CH_3COOC_2H_5}_{\text{Ethyl acetate.}} + NaOH = CH_3COONa + C_2H_5OH.$$

Let a and b represent the original concentrations of $CH_3COOC_2H_5$ and NaOH respectively, and x the amount of each (*i.e.* in gramme-molecules) transformed during the time t. Then by the law of mass action

$$\frac{dx}{dt} = k(a-x)(b-x),$$

$$\therefore \quad \frac{dx}{(a-x)(b-x)} = kdt.$$

$$\therefore \quad kt = \int \frac{dx}{(a-x)(b-x)}.$$

Resolving $\frac{dx}{(a-x)(b-x)}$ into partial fractions (see p. 30), we find that

$$\frac{dx}{(a-x)(b-x)} = \frac{dx}{(a-b)}\left\{\frac{1}{b-x} - \frac{1}{a-x}\right\}$$

$$\therefore \quad kt = \frac{1}{a-b}\left[\int\frac{dx}{b-x} - \int\frac{dx}{a-x}\right]$$

$$= \frac{1}{a-b}\left\{\log\frac{b}{b-x} - \log\frac{a}{a-x}\right\} + \mathrm{C}$$

$$= \frac{1}{a-b}\left\{\begin{array}{c}\log b - \log(b-x) - \log a \\ + \log(a-x)\end{array}\right\} + \mathrm{C}$$

$$= \frac{1}{a-b}\left\{\log\frac{b}{a} + \log\frac{(a-x)}{(b-x)}\right\} + \mathrm{C}$$

$$= \frac{1}{a-b}\log\frac{b}{a}\frac{(a-x)}{(b-x)} + \mathrm{C},$$

whence $$k = \frac{1}{t(a-b)}\log\frac{b}{a}\frac{(a-x)}{(b-x)} + \mathrm{C}.$$

Supposing we start with equal numbers of molecules of the reacting substances, we then have $a = b$, and our differential equation becomes

$$\frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{k(a-x)^2}.$$

This yields on integration

$$k = \frac{1}{at}\frac{x}{(a-x)}.$$

Hence if the ratio $\frac{x}{at(a-x)}$ is constant at every stage of the reaction, the transformation takes place as a bimolecular reaction. The amount x transformed during a time t in such a reaction is easily seen to be given by

$$x = \frac{ka^2t}{1+kat}$$

and

$$a - x = \frac{a}{1+kat}.$$

EXAMPLES.

(1) The following figures have been obtained for the values of $a - x$ at different times t in the case of the hydrolysis of ethyl acetate by means of NaOH. Prove that the reaction proceeds as a bimolecular one. The reaction was started with equivalent quantities of the reacting substances.

t (mins.)	$a - x$
0	8·04
4	5·30
6	4·58
8	3·91
10	3·51
12	3·12

This gives $a = 8{\cdot}04$, since at $t = 0$, $x = 0$.

$\therefore$ x at any time $= 8{\cdot}04$ less the corresponding value of $a - x$. Thus at $t = 4$, $x = 8{\cdot}04 - 5{\cdot}30 = 2{\cdot}74$, and so on.

In order to prove that the reaction is a bimolecular one it will be sufficient if we prove that by putting $k = \frac{1}{at}\frac{x}{(a-x)}$ and substituting the various corresponding values of x and t in this equation, we get uniform results for the value of k.

By doing so we find the following results:—

$$k_1 = \frac{1}{8{\cdot}04 \times 4} \times \frac{2{\cdot}74}{5{\cdot}30} = 0{\cdot}0161.$$

$$k_2 = \frac{1}{8{\cdot}04 \times 6} \times \frac{3{\cdot}46}{4{\cdot}58} = 0{\cdot}0157.$$

$$k_3 = \frac{1}{8{\cdot}04 \times 8} \times \frac{4{\cdot}13}{3{\cdot}91} = 0{\cdot}0164.$$

$$k_4 = \frac{1}{8{\cdot}04 \times 10} \times \frac{4{\cdot}53}{3{\cdot}51} = 0{\cdot}0161.$$

$$k_5 = \frac{1}{8{\cdot}04 \times 12} \times \frac{4{\cdot}92}{3{\cdot}12} = 0{\cdot}0163.$$

Hence, within the limits of experimental error the value of k is uniform, and therefore the reaction proceeds as a bimolecular reaction (see further p. 369 *et seq.*).

(2) The following figures have been obtained by Senter in the study of the decomposition of H_2O_2 by hæmase (blood-catalase). The reaction was followed by withdrawing a portion of the solution at fixed intervals (t), stopping the catalysis by means of excess of H_2SO_4, and titrating with $KMnO_4$ solution.

$$(2KMnO_4 + 3H_2SO_4 + 5H_2O_2 = K_2SO_4 + 2MnSO_4 + 5O_2 + 8H_2O).$$

t (mins.).	$a - x$ (c.c. $KMnO_4$ solution).	x (c.c. $KMnO_4$ solution).	$k = \frac{2 \cdot 3}{t} \log \frac{a}{a-x}$.
0	46·1	0	. .
5	37·1	9·0	0·0435
10	29·8	16·3	0·0438
20	19·6	26·5	0·0429
30	12·3	33·8	0·0440
50	5·0	41·1	0·0444

Prove that the reaction proceeds as a unimolecular one.

Putting $k = \frac{2 \cdot 3}{t} \log \frac{a}{a-x}$, the values of k are those given in the last column, *i.e.* are uniform. Therefore the reaction is unimolecular.

(3) Madsen and Walbum studied the destruction of coli-agglutinin by means of trypsin, at 35·6° C., in the following manner: They observed the quantity of agglutinin (q) which must be added to a suspension of *Bacillus coli* to obtain a given agglutination in a given time. The strength (S) of agglutinin added to the suspension is, of course, inversely proportional to q.

The following values of S were found for the corresponding values of t. Show that these values agree with those one would expect to find in the case of a bimolecular reaction.

Time (hours).	S (observed) $(a - x)$.	S (calculated on the supposition that reaction is bimolecular, viz. $S = a - x = \frac{a}{1 + kat}$).
0	1000	1000
0·5	775	747
1	610	595
2·25	389	395
3	280	329
4·17	259	261
5	233	227
6	189	197
8	149	155
10	140	128
12	108	109
25	59	56

For a bimolecular reaction, $\dfrac{x}{at(a-x)}$ should be constant $= k$. Here $a = 1000$, and $(a-x)$ at any moment t is the value of S at that moment, so that $x_t = 1000 - S_t$. Substituting these various values of x, $(a-x)$ and t in the foregoing expression we get the values of k for different values of t. These will be found fairly constant (lying between 58×10^{-5} and 85×10^{-5}, with an average of 68×10^{-5}). (*E.g.* at $t = 5$, we have

$$\frac{x}{at(a-x)} = \frac{1000 - 233}{5000 \times 233} = \frac{767}{1,165,000} = 66 \times 10^{-5} = k).$$

Hence it is probable that the disintegration of agglutinin proceeds as a bimolecular reaction.

Knowing the value of k we can now find the theoretical values of S, or $(a-x)$, for different values of t, by means of the equation $a - x = \dfrac{a}{1+kat}$. These are given in the 3rd column of the table, from which it is seen that the agreement between the observed and calculated values is very reasonably good. (*E.g.* at $t = 10$, we have

$$\frac{a}{1+kat} = \frac{1000}{1 + 68 \times 10^{-5} \times 10,000} = \frac{1000}{7 \cdot 8} = 128,$$

as against the observed value 140. At $t = 12$ the calculated value of S is 109, as against the observed value of 108, and so on.)

Corollary.—Since for a bimolecular reaction

$$k = \frac{x}{at(a-x)},$$

$$\therefore \quad t = \frac{x}{ka(a-x)}.$$

Hence, when the reaction is half complete, *i.e.* when $x = \dfrac{a}{2}$,

we get
$$t = \frac{\dfrac{a}{2}}{ka \times \dfrac{a}{2}} = \frac{1}{ka}.$$

In other words, **the time taken in a bimolecular reaction to change half the original quantity is inversely proportional to the initial concentration** a, **i.e.** $t_{\frac{1}{2}} \propto \dfrac{1}{a}$.

Similarly, **in the case of an n-molecular reaction,** in which the original concentrations of the reacting substances are the same, it can be easily shown by integrating the equation $\dfrac{dx}{dt} = k(a-x)^n$, that **the time taken to change half the original quantity is proportional to** $\dfrac{1}{a^{n-1}}$, **i.e. inversely proportional to** a^{n-1} (see Chapter XXII., p. 372 *et seq.*).

Chemical Equilibrium.—If in the *reversible reaction*

$$CH_3COOH + C_2H_5OH \rightleftharpoons CH_3COOC_2H_5 + H_2O,$$

k_1 represents the velocity constant of esterification, and k_2 represents the velocity constant of the opposite reaction, then we have

$$v_1 = k_1 C_{acid} \times C_{alcohol}$$

and
$$v_2 = k_2 C_{ester} \times C_{water}$$

where v_1 and v_2 are the reaction velocities in the two directions.

When equilibrium exists

$$v_1 = v_2$$

i.e.
$$k_1 C_{acid} \times C_{alcohol} = k_2 C_{ester} \times C_{water}$$

wherefore
$$\frac{k_1}{k_2} = \frac{C_{ester} \times C_{water}}{C_{acid} \times C_{alcohol}} = \text{Constant } (k).$$

In other words, whatever the original concentrations of the reacting substances, the ratio $\frac{C_{ester} \times C_{water}}{C_{acid} \times C_{alcohol}}$, at the moment of equilibrium, always has the same value k.

Evaluation of k:—

Now, experiment shows that if one starts with equal numbers of gramme-molecules of acid and alcohol, then equilibrium is reached when $\frac{2}{3}$ of the original amount of the substances present are decomposed. But when equal numbers of gramme-molecules are present, then $C_{acid} = C_{alcohol} = C$.

$$\therefore \quad \frac{k_1}{k_2}, \quad \text{or} \quad k, = \frac{\frac{2}{3}C \times \frac{2}{3}C}{\frac{1}{3}C \times \frac{1}{3}C} = 4.$$

From this value of k one can always predict when equilibrium will be established if one starts with any known amounts of acid and alcohol.

EXAMPLES.

(1) One gramme-molecule of acetic acid is mixed with a gramme-molecules of alcohol. Find the number of gramme-molecules, n, of acid decomposed when equilibrium has been established.

Since original $C_{acid} = 1$,

$$\therefore \quad C_{acid} \text{ when equilibrium exists} = 1 - n,$$

and
$$C_{alcohol} \quad ,, \quad ,, \quad ,, \quad = a - n,$$
$$C_{ester} \quad ,, \quad ,, \quad ,, \quad = n,$$
$$C_{water} \quad ,, \quad ,, \quad ,, \quad = n.$$

$$\therefore \quad k = 4 = \frac{n \times n}{(1-n)(a-n)} = \frac{n^2}{(1-n)(a-n)},$$

whence $n = \frac{2}{3}(a + 1 - \sqrt{a^2 - a + 1})$ (see p. 22).

[The full solution is, of course, $n = \frac{2}{3}(a+1 \pm \sqrt{a^2-a+1})$, but the student will see that $n = \frac{2}{3}(a+1+\sqrt{a^2-a+1})$ is an impossible answer, because it makes n greater than the initial number of gramme-molecules of acid.]

(2) Berthelot and Péan de St. Giles tested the above formula by mixing one gramme-molecule of acetic acid with the following numbers of gramme-molecules (a) of alcohol, and after the establishment of equilibrium found the following values of n:—

—	a.	n.
(1)	0·05	0·05
(2)	0·08	0·078
(3)	0·18	0·171
(4)	0·28	0·226
(5)	1	0·665
(6)	8	0·966

Find whether the calculated results agree with the observed results.

From the formula

$$n = \tfrac{2}{3}(a+1-\sqrt{a^2-a+1}) \quad \text{we get}$$

(1) $$n_1 = \tfrac{2}{3}(1{\cdot}05-\sqrt{0{\cdot}9525}) = 0{\cdot}05.$$

(2) $$n_2 = \tfrac{2}{3}(1{\cdot}08-\sqrt{0{\cdot}9264}) = 0{\cdot}078.$$

(3) $$n_3 = \tfrac{2}{3}(1{\cdot}18-\sqrt{0{\cdot}8524}) = 0{\cdot}171.$$

Similarly, $n_4 = 0{\cdot}26$; $n_5 = 0{\cdot}67$; and $n_6 = 0{\cdot}96$.

Application of the Law of Mass Action to the Dissociation of Oxyhæmoglobin.

The equation for this reversible reaction is

$$Hb + O_2 \rightleftharpoons HbO_2.$$

Put $\quad$ Hb concentration $= C_{(Hb)}$

$\quad O_2 \quad$,, $\quad = C_{(O_2)}$

and $\quad HbO_2 \quad$,, $\quad = C_{(HbO_2)}$

We then have, when equilibrium is established,

$$k_1 C_{(Hb)} \cdot C_{(O_2)} = k_2 C_{(HbO_2)}$$

$$\therefore \quad \frac{k_1}{k_2} = k = \frac{C_{(HbO_2)}}{C_{(Hb)} \cdot C_{(O_2)}}$$

Now, percentage saturation of Hb with oxygen is obviously

$$\frac{100 C_{(HbO_2)}}{C_{(Hb)} + C_{(HbO_2)}}$$

$\therefore$ If we designate the oxygen concentration ($C_{(O_2)}$) by x, and the percentage saturation of Hb with oxygen by y,

we get from $k = \dfrac{C_{(HbO_2)}}{C_{(Hb)} \cdot C_{(O_2)}}, \quad kx = \dfrac{C_{(HbO_2)}}{C_{(Hb)}} \quad . \quad . \quad (1)$

Also from $y = \dfrac{100\ C_{(HbO_2)}}{C_{(Hb)} + C_{(HbO_2)}}$

we get $\dfrac{100}{y} = \dfrac{C_{(Hb)} + C_{(HbO_2)}}{C_{(HbO_2)}}$,

$$\therefore \quad \frac{100 - y}{y}\left(i.e.\ \frac{100}{y} - 1\right) = \frac{C_{(Hb)}}{C_{(HbO_2)}}\left(i.e.\ \frac{C_{(Hb)} + C_{(HbO_2)}}{C_{(HbO_2)}} - 1\right),$$

$$\therefore \quad \frac{y}{100 - y} = \frac{C_{(HbO_2)}}{C_{(Hb)}} \quad . \quad . \quad . \quad . \quad (2)$$

But from (1), $\dfrac{C_{(HbO_2)}}{C_{(Hb)}} = kx$;

$$\therefore \quad \frac{y}{100 - y} = kx;$$

$$\therefore \quad \frac{100 - y}{y} = \frac{1}{kx};$$

$$\therefore \quad \frac{100}{y} = \frac{1 + kx}{kx}$$

(*i.e.* by adding 1 to each side of the equation);

$$\therefore \quad y = \frac{100kx}{1 + kx}.$$

The author is indebted to Dr. W. A. M. Smart for the method of deriving this formula.

The general equation of a reversible reaction may be written in the form:

$$aA + bB + cC + \ . \ . \ . \rightleftharpoons a_1A_1 + b_1B_1 + c_1C_1 + \ . \ . \ .$$

(where a molecules * of A react with b molecules of B, c molecules of C, etc., to form a_1 molecules of A_1, b_1 molecules of B_1, c_1 molecules of C_1, etc.).

$$\therefore \quad k_1 C_A^a \cdot C_B^b \cdot C_C^c \ . \ . \ . = k_2 C_{A_1}^{a_1} \cdot C_{B_1}^{b_1} \cdot C_{C_1}^{c_1} \ . \ . \ .$$

where C_A, C_B, etc., have the same meaning as similar symbols on p. 257.

$\therefore$ When equilibrium is established we have

$$k = \frac{k_1}{k_2} = \frac{C_{A_1}^{a_1} \cdot C_{B_1}^{b_1} \cdot C_{C_1}^{c_1} \ . \ . \ .}{C_A^a \cdot C_B^b \cdot C_C^c \ . \ . \ .}$$

* "Gramme-molecules" and "molecules" are used as synonymous terms.

From this it follows that, if instead of taking one molecule of Hb and combining it with one molecule of O_2, we take n molecules of each, we get for our equilibrium constant

$$k = \frac{C^n_{(HbO_2)}}{C^n_{(Hb)} \cdot C^n_{(O_2)}}, \quad \therefore \quad kx^n = \frac{C^n_{(HbO_2)}}{C^n_{(Hb)}}.$$

Also, $$y = \frac{100\ C^n_{(HbO_2)}}{C^n_{(Hb)} + C^n_{(HbO_2)}};$$

$$\therefore \quad y = \frac{100kx^n}{1 + kx^n},$$

which is A. V. Hills' equation for the dissociation of oxyhæmoglobin.

CHAPTER XVI.

THERMODYNAMIC CONSIDERATIONS AND THEIR BIOLOGICAL APPLICATIONS.

Thermodynamic Equations.—This is a convenient place to take up the mathematical consideration of a few points in connection with the general gas equation $PV = RT$, where P = pressure of an ideal gas, V = volume, T = absolute temperature, and R = a constant called the *gas constant*. This equation, as we shall see, is of great importance in the study of numerous problems of biological interest.

(1) **To Find the Numerical Value of R.**—

Since $\quad PV = RT,$

$$\therefore \quad R = \frac{PV}{T} \text{ for any values of P, V and T.}$$

If we take a gramme-molecule of any gas at 0° C. and 76 cm. pressure (*i.e.* atmospheric pressure), we have

$P = 76 \times 13{\cdot}6 = 1033$ grammes per sq. cm. (since the sp. gr. of mercury $= 13{\cdot}6$),

$V = 22{\cdot}4$ litres $= 22{,}400$ c.c. (since by Avogadro's law a gramme-molecule of any gas at N.T.P. occupies 22·4 litres),

and $\quad T = 273.$

$$\therefore \quad R = \frac{PV}{T} = \frac{1033 \times 22{,}400}{273} = 84{,}760 \text{ grm.-cm.}$$

$$= 0{\cdot}848 \text{ kilogram-metre.}$$

Also, since 42,640 grm.-cm. is equivalent to one calorie,

$$\therefore \quad R = \frac{84{,}760}{42{,}640} = 2 \text{ calories approximately (more exactly}$$

1·985 calories).

R can also be expressed in ergs (*i.e.* in c.g.s. units of work) or in volt-coulombs (*i.e.* units of electrical energy). Thus, P (pressure of one atmosphere) = 1033 grammes per sq. cm.

$= 1033 \times 981$ dynes per sq. cm. (p. 60)

$= 1{,}013{,}000$ dynes per sq. cm. approximately.

Now, 1 dyne acting through 1 cm. = 1 erg.

$$\therefore \quad R = \frac{PV}{T} = \frac{1{,}013{,}000 \times 22{,}400}{273} = 8\cdot3 \times 10^7 \text{ ergs}$$

$$= 8\cdot3 \text{ volt-coulombs or joules}$$

(one volt-coulomb being equivalent to 10^7 ergs or one joule).

(2) **Work done by a Gas during Isothermal Expansion.**—By isothermal expansion is meant the expansion which takes place at constant temperature, as when the gas is allowed to expand inside a cylinder (with a movable piston) (fig. 107) which is enclosed in a thermostat supplying heat to the gas to keep its temperature constant. Let the original volume of the gas be V, when the piston is at AA. Let the gas expand so as to shift the piston through the infinitesimally small distance AB. The volume of the gas will then become $V + dV$ (where dV = the infinitesimally small increase in V). Let dW represent the infinitesimally small amount of work done by the gas during this expansion from V to $V + dV$.

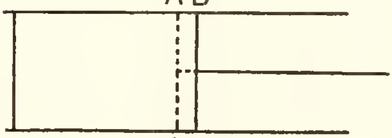


FIG. 107.

Then, since the pressure of the gas may be considered to have remained constant during this infinitesimally small increase in volume, and since also work is measured by force or pressure multiplied by the space through which it acts, we have

$$dW = PdV.$$

We now have to find the amount of work done **by** the gas when it expands by a **finite** amount, *i.e.* from volume V_1 to a volume V_2. This we do by integrating, between limits, as follows:—

$$\int dW = \int_{V_1}^{V_2} PdV.$$

Now whilst $$\int dW = W,$$

$$\int_{V_1}^{V_2} PdV \text{ is } \textbf{not} \text{ equal to } P\int_{V_1}^{V_2} dV, \text{ or } P(V_2 - V_1),$$

because, although P was practically constant during the minute increase of volume dV, it keeps on varying during the finite increase in volume from V_1 to V_2. But remembering the gas equation

$$PV = RT$$

(where T is the absolute temperature, which under isothermal conditions remains constant, and R is the gas constant), we can put

$$P = \frac{RT}{V}.$$

Hence our differential equation becomes

$$dW = RT\frac{dV}{V}.$$

$$\therefore \quad \int dW = \int_{V_1}^{V_2} RT\frac{dV}{V},$$

i.e.

$$W = RT\int_{V_1}^{V_2} \frac{dV}{V},$$

or

$$\mathbf{W = RT \log_e \frac{V_2}{V_1} = 2{\cdot}3\ RT \log_{10} \frac{V_2}{V_1}}.$$

The work done in *compressing* the gas is thus equal to $2{\cdot}3\ RT \log_{10} \frac{V_1}{V_2}$.

Application of the foregoing Equation to Solutions.—Since it has been shown experimentally that electrolytes in solution behave like gases, and since the concentration of a substance is inversely proportional to the volume of the solution, we have

$$\frac{c_1}{c_2} = \frac{V_2}{V_1}.$$

$\therefore$ The amount of work that must be done **on** a solution to increase its concentration from c_1 to c_2 is given by

$$W = 2{\cdot}3RT \log_{10} \frac{c_2}{c_1}.$$

Further, since the depression in freezing-point (Δ) of a solution is proportional to the concentration—

$\therefore$ The amount of work that must be done on a solution to change the freezing-point from Δ_1 to Δ_2 is

$$W = 2{\cdot}3RT \log_{10} \frac{\Delta_2}{\Delta_1}.$$

This is an equation of fundamental importance in biochemistry.

Note.—The gas constant has the following values, viz.:

$$\left.\begin{array}{ll} R & = 2 \text{ gramme-calories} \\ & = 0{\cdot}85 \text{ kilogram-metre} \\ & = 0{\cdot}082 \text{ litre-atmosphere} \end{array}\right\} \text{(see p. 269).}$$

Example.—The freezing-point of blood is $-0{\cdot}56°$ C.; that of urine is $-1{\cdot}85°$ C. Assuming a healthy adult to pass 1·5 litres of urine a day, calculate the work done by the kidneys in a day.

The work done by the kidneys to concentrate **one litre** of a glomerular filtrate which has $\Delta_1 = -0{\cdot}56°$ to a urine for which $\Delta_2 = -1{\cdot}85°$, at body temperature 37° C., is given by

$$W = 2{\cdot}3R(273+37)\log_{10}\frac{1{\cdot}85}{0{\cdot}56}.$$

Taking $R = 0{\cdot}85$ kilogram-metre we get

$$\begin{aligned} W &= 2{\cdot}3\times 0{\cdot}85\times 310\,(\log 1{\cdot}85 - \log 0{\cdot}56) \\ &= 606(0{\cdot}2672 - \bar{1}{\cdot}7482) \\ &= 606\times 0{\cdot}519 \\ &= 315 \text{ kilogram-metres.} \end{aligned}$$

∴ To concentrate **1·5 litres** of the glomerular filtrate, the amount of work necessary $= 1{\cdot}5\times 315$

$= 472{\cdot}5$ kilogram-metres.

Note.—This calculation of the work done by the kidneys takes into account only the total concentration of the blood and urine, and is meant merely as an illustration of the application of the isothermal expansion formula to physiology. In order to calculate the actual work of the kidneys, account must be taken of the change of concentration of each of the urinary constituents. (See Cushing, "*The Secretion of Urine*"; and Bayliss, "*The Principles of General Physiology.*")

Application of the Isothermal Expansion Equation to Concentration Cells.—If we put two electrodes of the *same* univalent metal, such as silver, into two solutions of the *same* salt, but of different concentrations, c_1 and c_2, and join them up, we get a battery known as a "concentration cell," and physico-chemical considerations, similar to those in the preceding paragraph, lead to the equation

$$E = \frac{2{\cdot}3RT}{F}\log_{10}\frac{c_2}{c_1},$$

where E is the E.M.F. (electromotive force) in volts,
R is the gas constant expressed in coulomb-volts
$= 8{\cdot}36$ (since 1 calorie $= 4{\cdot}18$ coulomb-volt).
F is the amount of electricity transported by one gramme-ion of the metal = approx. 96,500 coulombs = 1 faraday.

∴ At room temperature, *i.e.* at 17° C.,

$$E \doteq \frac{2{\cdot}3\times 8{\cdot}4\times 290}{96{,}500}\log_{10}\frac{c_2}{c_1} \text{ volts}$$

$$= 0{\cdot}0002\times 290\log_{10}\frac{c_2}{c_1} = 0{\cdot}058\log_{10}\frac{c_2}{c_1} \text{ volts.}$$

This formula must also be multiplied by $\dfrac{u-v}{u+v}$, where u and v

are the migration velocities of the ions of the salt or acid in question (because it is the ions that carry the electrical charge, so that for one gramme-ion, $\frac{u}{u+v}$ gramme-ion of say H-ions or other positive ions have travelled in one direction and $\frac{v}{u+v}$ gramme-ion of the other (negative) ions have travelled in the opposite direction).

$\therefore$ Finally, $E = 0{\cdot}058\frac{u-v}{u+v}\log_{10}\frac{c_2}{c_1}$ volts (at room temperature).

At any other temperature, t, the formula is

$$E = 0{\cdot}0002 \times (273+t)\frac{u-v}{u+v}\log_{10}\frac{c_2}{c_1}.$$

Therefore, knowing c_1 and c_2 for any electrolyte (the rates of migration of the ions of which are known), the value of E can be calculated for any temperature t. Such calculated values have been found to agree well with experimental measurements of E.

From this equation it follows that if the salt used has ions which migrate at approximately the same velocity, E may be too small to be measured.

If the electrode be of *hydrogen* (*e.g.* platinum saturated with hydrogen), this equation enables one to calculate the H-ion concentration of a given solution. (See further, books on Physical Chemistry.)

(3) The Relation between the Two Specific Heats of Gases.

(i) *The Value of the Arithmetical Difference* $S_p - S_v$.

(*a*) If we raise 1 gramme of gas from 0° to 1° C., **keeping the volume constant,** then the quantity of heat required is called the **specific heat at constant volume** and is designated $\mathbf{S}_v$.

(*b*) If the same quantity of the gas is raised from 0° to 1° C., but is allowed to expand **at constant pressure,** then the quantity of heat required for the purpose is called the **specific heat at constant pressure,** and is designated $\mathbf{S}_p$.

It is clear that $S_p > S_v$, since if the gas is allowed to expand from V_1 to V_2 at the constant pressure P, it does an amount of work which is measured by $P(V_2 - V_1)$, and hence the quantity of heat supplied must be sufficient not only to raise the temperature of the gas by 1° C. but also to make up for the cooling entailed by the work of expansion.

$$\therefore \qquad S_p - S_v = P(V_2 - V_1).$$

But $$PV_1 = RT_1,$$
and $$PV_2 = RT_2;$$
$$\therefore \quad P(V_2 - V_1) = R(T_2 - T_1);$$
$\therefore$ when $$T_2 - T_1 = 1^\circ,$$
$$P(V_2 - V_1) = R.$$
$$\therefore \quad S_p - S_v = R.$$

If, instead of heating 1 gramme of the gas to 1° C., we heat 1 gramme-molecule, the amounts of heat required in each case are called the molecular heats at constant volume or pressure. In that case, the *difference between the two molecular heats* = 2 *calories* (which is the value of R for a gramme-molecule of gas).

(ii) *The Ratio* γ *between the Two Specific Heats*

$$\left(\text{i.e. } \textit{the value } \frac{S_p}{S_v}\right).$$

(*a*) Let dQ be a minute amount of heat imparted to a gramme of gas at **constant volume.** This will go entirely to raise the temperature of the gas by dT.

$$\therefore \quad dQ = S_v dT.$$

(*b*) If dQ be the minute quantity of heat imparted to a gramme of gas **at constant pressure,** then dQ is distributed into two forms of energy, viz.—(α) one portion goes to raise the temperature by dT; (β) another portion goes to perform work of expansion.

The portion under (α) is represented by $S_v dT$ and the portion under (β) is represented by PdV.

$$\therefore \quad dQ = S_v dT + PdV.$$
But $$PV = RT.$$

$\therefore$ When P, V and T are variable, we get by differentiation

$$\frac{d(PV)}{dT} = R,$$

i.e. $$\frac{PdV}{dT} + \frac{VdP}{dT} = R \text{ (p. 173)}$$

$$\therefore \quad PdV + VdP = RdT$$

and $$\frac{PdV}{R} + \frac{VdP}{R} = dT.$$

But $$dQ = S_v dT + PdV.$$

$$\therefore \quad dQ = S_v\left(\frac{PdV}{R} + \frac{VdP}{R}\right) + PdV$$
$$= PdV\left(\frac{S_v + R}{R}\right) + \frac{S_vVdP}{R}$$
$$= \frac{S_pPdV}{R} + \frac{S_vVdP}{R} \text{ (see p. 274).}$$

If the cylinder containing the gas be now surrounded by a non-conducting envelope, so as to prevent heat from entering or leaving the cylinder, and P, V and T be allowed to change **adiabatically** (*i.e.* without transfer of heat to or from the gas), then $dQ = 0$.

$$\therefore \quad S_pPdV + S_vVdP = 0,$$

or
$$\frac{S_p}{S_v}PdV + VdP = 0,$$

i.e.
$$\gamma PdV + VdP = 0,$$

or, dividing by PV,
$$\gamma\frac{dV}{V} + \frac{dP}{P} = 0.$$

$$\therefore \quad \gamma\int_{V_1}^{V_2}\frac{dV}{V} + \int_{P_1}^{P_2}\frac{dP}{P} = 0,$$

i.e.
$$\gamma \log_e \frac{V_2}{V_1} + \log_e \frac{P_2}{P_1} = 0,$$

or
$$\gamma \log_e \frac{V_2}{V_1} = \log_e \frac{P_1}{P_2}.$$

$$\therefore \quad \left(\frac{V_2}{V_1}\right)^\gamma = \frac{P_1}{P_2} \text{ or } PV^\gamma = C \text{ (constant).}$$

(4) **Law of Adiabatic Expansion** (*i.e.* expansion without transfer of heat to or from the gas).

Since $dQ = S_vdT + PdV$ (see p. 274),
and also $PdV = RdT - VdP$ (p. 274),

$$\therefore \quad dQ = S_vdT + RdT - VdP$$
$$= (S_v + R)dT - VdP$$
$$= S_pdT - VdP \text{ (see p. 274)}$$
$$= S_pdT - RT\frac{dP}{P} \text{ (since } PV = RT).$$

$\therefore$ For an adiabatic expression, when $dQ = 0$,

$$S_pdT = RT\frac{dP}{P}.$$

But $$R = S_p - S_v \text{ (see p. 274).}$$

$$\therefore \quad S_p dT = (S_p - S_v)T\frac{dP}{P},$$

or $$\frac{S_p}{S_p - S_v}\frac{dT}{T} = \frac{dP}{P}.$$

$$\therefore \quad \frac{S_p}{S_p - S_v}\int_{T_1}^{T_2}\frac{dT}{T} = \int_{P_1}^{P_2}\frac{dP}{P},$$

i.e. $$\frac{S_p}{S_p - S_v}\log_e\frac{T_2}{T_1} = \log_e\frac{P_2}{P_1},$$

or $$\frac{S_p/S_v}{\frac{S_p}{S_v} - 1}\log_e\frac{T_2}{T_1} = \log_e\frac{P_2}{P_1},$$

i.e. $$\frac{\gamma}{\gamma - 1}\log_e\frac{T_2}{T_1} = \log_e\frac{P_2}{P_1}.$$

$$\therefore \quad \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \frac{P_2}{P_1},$$

or $$\left(\frac{T_2}{T_1}\right)^{\gamma} = \left(\frac{P_2}{P_1}\right)^{\gamma-1}$$

But we have seen (p. 275) that

$$\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{\gamma}$$

$$\therefore \quad \left(\frac{T_2}{T_1}\right)^{\gamma} = \left\{\left(\frac{V_1}{V_2}\right)^{\gamma}\right\}^{\gamma-1}$$

or $$\frac{\mathbf{T_2}}{\mathbf{T_1}} = \left(\frac{\mathbf{V_1}}{\mathbf{V_2}}\right)^{\gamma-1}$$

(5) **Efficiency of a Heat Engine.**—By the efficiency of a heat engine is meant the proportion of the heat developed by the engine which is transformed into mechanical work. Considerations based upon the second law of thermodynamics lead to the conclusion that the *maximum efficiency* is obtainable only from a *reversible engine, i.e.* an engine which, "after converting a certain fraction of the heat into work, will return to its original state in every respect if made to act backwards step by step." To estimate the maximum efficiency of an engine, we must consider *Carnot's reversible cycle.*

Carnot's Cycle (fig. 108).—(i) Let a gramme-molecule of gas of volume V_1, pressure P_1 and temperature T_1 (represented in the diagram by the point A) be allowed to *expand* **isothermally** to the point B (V_2, P_2).

Then, if during the process Q_1 units of heat have been **absorbed from** the thermostat, we have

$$Q_1 = \int_{V_1}^{V_2} P dV = RT_1 \log_e \frac{V_2}{V_1}$$

(p. 271).

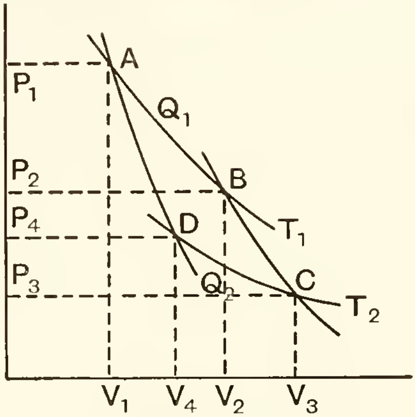


FIG. 108.—Carnot's Cycle.

(ii) Now let the gas *expand* **adiabatically,** *i.e.* without allowing it to gain or lose heat, to C (V_3, P_3). The temperature of the gas will fall from T_1 to T_2, and we have

$$\frac{T_1}{T_2} = \left(\frac{V_3}{V_2}\right)^{\gamma - 1} \text{(see p. 276).}$$

(iii) Now let the gas be *compressed* **isothermally** at temperature T_2 to D (V_4, P_4). Then if Q_2 is the amount of heat *evolved* from the gas **into** the thermostat, we have

$$-Q_2 = \int_{V_3}^{V_4} P dV = RT_2 \log_e \frac{V_4}{V_3}$$

(heat evolved being of opposite sign to that absorbed),

$$\text{or} \quad Q_2 = RT_2 \log_e \frac{V_3}{V_4}$$

$$\left(\text{since} \quad \log \frac{V_4}{V_3} = -\log \frac{V_3}{V_4}\right).$$

$$\therefore \quad \frac{Q_1}{Q_2} = \frac{T_1 \log_e \frac{V_2}{V_1}}{T_2 \log_e \frac{V_3}{V_4}}.$$

(iv) Finally, let the gas be *compressed* **adiabatically** from D (V_4, P_4) to A (V_1, P_1), *i.e.* back again to its original volume, pressure and temperature. The temperature will rise from T_2 to T_1, and we have

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_4}\right)^{\gamma-1} \text{(p. 276)},$$

or
$$\frac{T_1}{T_2} = \left(\frac{V_4}{V_1}\right)^{\gamma-1}$$

But
$$\frac{T_1}{T_2} = \left(\frac{V_3}{V_2}\right)^{\gamma-1} \text{[by (ii), on p. 277]}.$$

$$\therefore \quad \frac{V_3}{V_2} = \frac{V_4}{V_1},$$

or
$$\frac{V_2}{V_1} = \frac{V_3}{V_4}.$$

$\therefore$ Equation
$$\frac{Q_1}{Q_2} = \frac{T_1 \log_e \frac{V_2}{V_1}}{T_2 \log_e \frac{V_3}{V_4}},$$

becomes
$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \quad \text{or} \quad \frac{Q_2}{Q_1} = \frac{T_2}{T_1},$$

or
$$\frac{Q_2}{Q_1} - 1 = \frac{T_2}{T_1} - 1;$$

or
$$\frac{Q_2 - Q_1}{Q_1} = \frac{T_2 - T_1}{T_1},$$

or
$$\frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}.$$

But sinee Q_1 is the amount of heat absorbed by the gas and Q_2 is the amount evolved,

$\therefore \quad \dfrac{Q_1 - Q_2}{Q_1}$ is the proportion of absorbed heat whieh is converted into work, and represents the *efficiency* of the system, which, being a reversible cycle, must possess the maximum possible efficiency.

$\therefore$ *Maximum efficiency* of a reversible engine (or any reversible proeess) $= \dfrac{T_1 - T_2}{T_1}$, where T_1 and T_2 are the absolute temperatures of the first and second isothermal operations.

E.g. If a steam engine receives steam at 140° C. and exhausts into the air at 100° C., then the efficiency is

$$\frac{140 - 100}{273 + 140} = \frac{40}{413} = 0{\cdot}097 = 9{\cdot}7 \text{ per cent.}$$

The ordinary efficiency of a steam engine is about 20 per cent.

Example.—Calorimetric experiments have shown that the efficiency of human muscle (*i.e.* the proportion of energy converted by muscle into work) is 20 per cent. Find whether it acts like a heat engine.

If it acts like a heat engine, then we have

$$\text{Efficiency} = \frac{T_1 - T_2}{T_1}.$$

If we take T_2 as the absolute body temperature

$$= 273 + 37 = 310,$$

then $$\frac{1}{5} = \frac{T_1 - 310}{T_1} \text{ (since } \frac{Q_1 - Q_2}{Q_1} = \frac{1}{5} \text{ by hypothesis).}$$

$$\therefore \quad T_1 = 5T_1 - 1550.$$

$$\therefore \quad 4T_1 = 1550,$$

or $$T_1 = 387{\cdot}5^\circ \text{ absolute}$$

$$= 114{\cdot}5^\circ \text{ C.}$$

Hence the body must show a fall of temperature during work from an internal temperature of, presumably, 114·5° C. to 37° C., which is absurd, as the body never attains such a temperature as 114·5° C.

If, on the other hand, we call the absolute body temperature T_1, then we have

$$\frac{1}{5} = \frac{310 - T_2}{310}$$

or $$62 = 310 - T_2.$$

$$\therefore \quad T_2 = 310 - 62 = 248^\circ \text{ absolute} = -25^\circ \text{ C.}$$

In other words, during contraction the temperature of the body must fall to −25° C., which is equally absurd. Hence muscle does **not** act like a heat engine.

Deductions from the Efficiency Equation.

(1) **Connection between the Latent Heat of Vaporisation and Change of Vapour Pressure with Temperature.**—When any fluid is heated the pressure of the saturated vapour varies with the temperature, *e.g.* in the case of water the vapour pressure at 10° C. = 9·179 mm. Hg, and at 100° C. = 760 mm. Hg. This means that if water be heated to 10° C. at 9·179 mm. pressure or to 100° C. at 760 mm., it will turn into vapour. In other words, the boiling-point of water at 9·179 mm. is 10° C., and at atmospheric pressure is 100° C. Now, suppose water at 99·5° C. (whose vapour pressure is 746·5 mm.) has the pressure reduced from atmospheric pressure, 760 mm., to 746·5 mm., then it will boil at 99·5° C. instead of at 100° C.

If we indicate the minute change of pressure by dP and the minute change of temperature by dT, then the work done by a gramme-molecule of water in boiling is

$$\frac{(V_2 - V_1)dP}{Q} = \frac{dT}{T}$$

(where V_2 = volume of a gramme-molecule of the vapour,
V_1 = ,, ,, ,, liquid,
Q = amount of heat required to change a gramme-molecule of the liquid **to the same volume** of vapour).

But as V_1 is negligible in comparison with V_2, the equation may be written

$$\frac{V_2 dP}{Q} = \frac{dT}{T}.$$

But since $PV_2 = RT, \quad \therefore \quad V_2 = \frac{RT}{P}$,

$$\therefore \qquad \frac{RT}{PQ}dP = \frac{dT}{T},$$

$$\therefore \qquad \frac{dP}{P} = \frac{QdT}{RT^2}$$

$$= \frac{Q}{2}\frac{dT}{T^2} \text{ [since R = 2 (see p. 269)].}$$

Now if we assume that Q remains constant between the limits of temperature T_1 and T_2, then

$$\int_{P_1}^{P_2}\frac{dP}{P} = \frac{Q}{2}\int_{T_1}^{T_2}\frac{dT}{T^2},$$

i.e.

$$\log_e \frac{P_2}{P_1} = \frac{Q}{2}\left(\frac{1}{T_1} - \frac{1}{T_2}\right),$$

or

$$2{\cdot}3 \log_{10} \frac{P_2}{P_1} = \frac{Q}{2}\frac{(T_2 - T_1)}{T_1 T_2}.$$

Hence, if we know the values of P_2 and P_1 for the known temperatures T_1 and T_2, Q may easily be calculated. If, now, we add to the value of Q thus found the amount of work done by the expansion of the vapour from V_1 to V_2, which is PV_2 (since V_1 is negligible) = 2T, the quantity so obtained is the mean latent heat of vaporisation of a gramme-molecule of liquid between T_1 and T_2.

(2) **Influence of Temperature upon the Velocity of a Chemical Reaction.**—Since substances in solution behave as if they were gases occupying the same volume as the solution, therefore P in the case of a solution (*i.e.* the osmotic pressure) is proportional to the concentration of the solution (*i.e.* inversely to the volume).

But since by the law of mass action the reaction velocity is proportional to the concentration, therefore P is proportional to the reaction velocity at the temperature in question.

Hence, if reaction velocity at $T_1 = K_1$,

and ,, ,, $T_2 = K_2$,

we have $$\log_e \frac{K_2}{K_1} = \frac{Q}{2}\,\frac{(T_2 - T_1)}{T_1 T_2},$$

or $$\frac{K_2}{K_1} = e^{\frac{Q}{2}\frac{(T_2-T_1)}{T_1T_2}} \quad \text{(Van't Hoff-Arrhenius law).}$$

Here Q is the amount of heat evolved in the reaction.

(For numerical examples, see p. 282 *et seq.*)

Note (1).—Since $\frac{1}{P}$ is the same as $\frac{d \log_e P}{dP}$, therefore the equation $\frac{dP}{P} = \frac{QdT}{RT^2}$ (on p. 280) may be written as $\frac{d \log_e P}{dT} = \frac{Q}{RT^2}$.

And as this equation enables us to calculate the influence of temperature upon equilibrium at **constant volume,** it was called by Van't Hoff the **isochore** equation in contradistinction to the mass action equation, which deals with the influence of change of concentration at **constant temperature,** and is therefore called the reaction **isotherm.**

Note (2).—It follows from the equation $\frac{K_2}{K_1} = e^{\frac{Q}{2}\frac{(T_2-T_1)}{(T_1T_2)}}$ that when $Q = 0$ then $K_2 = K_1$. This means that when the reaction is "thermo-neutral" (*i.e.* is not accompanied by any thermal change), then temperature has no influence upon its velocity. Thus, in the reaction

$$\underbrace{CH_3(CH_2)_2COOC_2H_5}_{\text{Ethyl butyrate}} + H_2O \rightleftharpoons \underbrace{CH_3(CH_2)_2COOH}_{\text{Butyric acid}} + C_2H_5OH,$$

the calorific value of $CH_3(CH_2)_2COOC_2H_5$ = 851·3 calories, and for $CH_3(CH_2)_2COOH$ and C_2H_5OH = 325·7 and 524·4 calories respectively, giving a total of 850·1 calories. Therefore there is practically no evolution or absorption of heat in this reaction, and, according to theory, temperature should have no influence upon its velocity. Experiment has confirmed theoretical expectation, for it has been found that in similar reactions the velocity at 10° C. is practically the same as that at 220° C.

Note (3).—If the temperature interval is sufficiently small to make the product T_1T_2 fairly constant, then $\frac{Q}{2}\frac{(T_2 - T_1)}{T_1T_2}$ may be written as $C(T_2 - T_1)$, where $C = \frac{Q}{2T_1T_2}$.

$$\therefore \quad \frac{K_2}{K_1} = e^{C(T_2-T_1)} \quad \text{or} \quad \log_{10}\frac{K_2}{K_1} = 0{\cdot}434C(T_2 - T_1)$$
$$= A(T_2 - T_1).$$

$$\therefore \quad \frac{K_2}{K_1} = 10^{A(T_2-T_1)}$$

If $T_2 - T_1 = 10$, this equation becomes

$$\frac{K_{t+10}}{K_t} = 10^{10A}$$

$\frac{K_{t+10}}{K_t}$ is called the **temperature coefficient** (see p. 15), and is generally found to lie between 2 and 3.

The efficiency equation has also been applied to calculate

(3) **The Heat of Solution of a Substance.**—In this case

$$2\cdot3 \log_{10} \frac{C_2}{C_1} = \frac{Q}{2} \frac{(T_2 - T_1)}{T_1 T_2},$$

where C_1 and C_2 are the concentrations of the saturated solutions at T_1 and T_2, and Q = heat of solution.

(4) **The Heat of Dissociation of Gases,** where a similar equation applies.

EXAMPLES.

(1) Madsen and his co-workers found the influence of temperature upon the velocity of hæmolysis, agglutination and precipitation to be the same as for other chemical reactions, viz. given by

$$\frac{K_2}{K_1} = e^{\frac{Q}{2}\frac{(T_2 - T_1)}{T_1 T_2}}$$

If the addition of 0·085 c.c. of an ammonia solution to 8 c.c. of a 1 per cent. solution of horse's erythrocytes produces hæmolysis in ten minutes at a temperature of 34·8° C., how many c.c. of the same ammonia solution added to the same quantity of erythrocytes will produce the same amount of hæmolysis in the same time at a temperature of 29·7° C.? [Q in this case = 26,760.]

As at 29·7° C. the velocity will be less than at 34·8° C., ∴ a greater amount of NH_3 solution will be required at 29·7° C. to produce the same result in the same time than at 34·8° C., and if x be the quantity required, then

$$\frac{K_{34\cdot8}}{K_{29\cdot7}} = \frac{x}{0\cdot085}.$$

$$\therefore \quad \frac{x}{0\cdot085} = e^{\frac{Q}{2}\frac{(T_{34\cdot8} - T_{29\cdot7})}{T_{34\cdot8}\,.\,T_{29\cdot7}}} = e^{13380 \times \frac{5\cdot1}{307\cdot8 \times 302\cdot7}}$$

$$= e^{0\cdot73}$$

$$\therefore \quad \log x - \log 0\cdot085 = 0\cdot73 \times 0\cdot4343 = 0\cdot317$$

$$\therefore \quad \log x - \bar{2}\cdot9294 = 0\cdot317, \text{ or } \log x = \bar{1}\cdot2464,$$

whence $x = 0\cdot176$ c.c.

(The observed result was found to be 0·17 c.c.)

Note.—It is easy to evaluate Q in any particular case, for T_1 and T_2 being known, and K_1 and K_2 being ascertainable for any reaction, Q is readily calculated. This is seen from the next example.

(2) Spohr found the velocity of reaction in the case of inversion of cane

sugar to be 9·67 at 25° C. and 139 at 45° C. Find what should be the theoretical velocity at 40° C.

Here we first have to find the value of Q from the data at 25° C. and 45° C. and then substitute it in the formula for K at 40° C.

Thus

$$\frac{K_{45}}{K_{25}} = e^{\frac{Q}{2}\,\frac{20}{298 \times 318}}$$

i.e.

$$\frac{139}{9 \cdot 67} = e^{\frac{10Q}{94764}}$$

$$\therefore \quad 2 \cdot 3 \log \frac{139}{9 \cdot 67} = \frac{10Q}{94764}$$

i.e.

$$2 \cdot 3(2 \cdot 1430 - 0 \cdot 9854) \times \frac{94{,}764}{10} = Q,$$

i.e.

$$Q = 2 \cdot 3 \times 1 \cdot 1576 \times 9476 \cdot 4$$
$$= 25{,}230.$$

$\therefore$ K at 40° C. is given by the equation

$$\frac{K_{40}}{K_{25}} = e^{\frac{25230}{2}\,\frac{15}{313 \times 298}}$$

or

$$2 \cdot 3 \log K_{40} - 2 \cdot 3 \log 9 \cdot 67 = 12615 \times \frac{15}{93274}$$

or

$$2 \cdot 3 \log K_{40} - 2 \cdot 266 = 2 \cdot 03.$$

$$\therefore \quad \log K_{40} = \frac{4 \cdot 296}{2 \cdot 3} = 1 \cdot 87.$$

$$\therefore \quad K_{40} = 74 \cdot 1.$$

(The observed value of K_{40} was 73·4.)

(3) Barcroft found the influence of temperature upon the dissociation velocity of oxyhæmoglobin to be of the usual character, viz. given by

$$\frac{K_2}{K_1} = e^{\frac{Q}{2}\,\frac{(T_2 - T_1)}{T_2 \,.\, T_1}}.$$

Calculate the molecular weight of hæmoglobin from the following data:—

(i) Percentages of hæmoglobin converted into oxyhæmoglobin at various temperatures when the pressure of oxygen was kept constant at 10 mm. Hg were found to be as follows:—

Temperature.	Percentage converted.
16° C.	92
24° C.	71
32° C.	37
38° C.	18
49° C.	6

(ii) Calorific value of 1 gramme hæmoglobin = 1·85 calories.

(iii) The dissociation is a reversible reaction as follows:—

Oxyhæmoglobin $\rightleftarrows$ hæmoglobin + oxygen.

If we call the concentration of oxyhæmoglobin $C_{(HbO)}$, the concentration of hæmoglobin $C_{(Hb)}$, and the concentration of oxygen $C_{(O)}$, then, by the law of mass action (see p. 266),

$$k_1 C_{(HbO)} = k_2 C_{(Hb)} \times C_{(O)} \quad . \quad . \quad . \quad . \quad (a)$$

Now, by Henry's law, the oxygen concentration $C_{(O)}$ is proportional to the pressure of the oxygen, p.

Therefore we can put the equation into the form

$$k_1 C_{(HbO)} = k_2 C_{(Hb)} \cdot p \quad . \quad . \quad . \quad . \quad (b)$$

(where k_2 has a different value from the k_2 in equation (a) above).

$$\therefore \quad \frac{k_1}{k_2} = \frac{C_{(Hb)} \cdot p}{C_{(HbO)}}$$

or

$$k = \frac{C_{(Hb)} \cdot p}{C_{(HbO)}}$$

(where k is the dissociation constant).

Hence if the pressure of oxygen (p) is kept constant at any temperature, then k is proportional to $\dfrac{C_{(Hb)}}{C_{(HbO)}}$.

$\therefore$ k_{16} is proportional to and may be put as equal to $\dfrac{C_{(Hb)}}{C_{(HbO)}}$ at 16° C., viz. 8/92.

k_{24}	,,	,,	,,	,,	24° C.,	,, 29/71.
k_{32}	,,	,,	,,	,,	32° C.,	,, 63/37.
k_{38}	,,	,,	,,	,,	38° C.,	,, 82/18.
k_{49}	,,	,,	,,	,,	49° C.,	,, 94/6.

Now

$$\frac{k_{24}}{k_{16}} = e^{\frac{Q}{2}\left(\frac{T_{24}-T_{16}}{T_{24} \cdot T_{16}}\right)}$$

$\therefore$

$$\tfrac{29}{71} \times \tfrac{92}{8} = e^{\frac{4Q}{297 \cdot 289}} = e^{\frac{Q}{21458}}$$

or

$$4{\cdot}7 = e^{\frac{Q}{21458}}$$

$\therefore$

$$\log 4{\cdot}7 = \frac{Q}{21458} \times 0{\cdot}4343.$$

$\therefore$

$$2{\cdot}3 \times 0{\cdot}6721 \times 21458 = Q = 33{,}180.$$

Similarly,

$$\frac{k_{32}}{k_{24}} \text{ or } \tfrac{63}{37} \times \tfrac{71}{29} = e^{\frac{Q}{22646}},$$

giving

$$Q = 32{,}330.$$

Also,

$$\frac{k_{38}}{k_{32}} \text{ or } 2{\cdot}675 = e^{\frac{Q}{31618}},$$

giving

$$Q = 31{,}120.$$

Lastly,

$$\frac{k_{49}}{k_{38}} \text{ or } 3{\cdot}439 = e^{\frac{Q}{18207}},$$

giving

$$Q = 22{,}470.$$

The mean of these four values of Q

$$= \frac{(33,180 + 32,330 + 31,120 + 22,470)}{4} = 29,800 \text{ (approx.)}.$$

Hence the amount of heat given off by the union of 1 gramme-molecule of hæmoglobin with oxygen $\doteqdot$ 29,800 calories. But from (ii) the union of 1 gramme of hæmoglobin with oxygen liberates 1·85 calories.

$$\therefore \quad \text{Molecular weight of hæmoglobin} = \frac{29,800}{1\cdot85} = 16,100,$$

which is practically identical with the known weight of hæmoglobin.

Note.—Amongst the vital processes which are accelerated by a rise of temperature in accordance with the foregoing law may be mentioned the following: the development of sea-urchin and other eggs (see p. 11); the conduction of impulse along a nerve, where experiment shows that

$$\frac{\text{Velocity of nerve impulse at } t_{n+10}}{\text{Velocity of nerve impulse at } t_n} = 2;$$

the action of drugs upon muscle; the rate of the heart-beat; phagocytosis; the reaction velocity of disinfection; the rhythm of the small intestine; respiration in plants, etc. Hence it follows that metabolism occurs more rapidly in cases of fever than normally. Similarly in the case of phagocytosis. But in this case it has been shown by Madsen and his collaborators that the temperature of the body at the time the corpuscles have been obtained is the optimum.

EXERCISES.

(1) A. J. Clark found the values of the rate of beat of the isolated auricle of the rabbit's heart at various temperatures to be as follows :—

Temperature (Centigrade) .	15	25	30	34	38
Rate (per minute) . .	25	64	82	120	170

Assuming the rate to vary with the temperature in accordance with the Van't Hoff-Arrhenius law, what is the mean value of Q?

[*Answer*, 14,835.]

(2) Assuming Q to have the value found in Question 1, calculate the theoretical rates of the auricle at the given temperatures.

Answer,

T° C. .	15	25	30	34	38
Rate .	25	59·5	89·6	123	174·3

(See W. M. Feldman and A. J. Clark, *The Lancet*, 1924, vol. i.)

CHAPTER XVII.

USE OF INTEGRAL CALCULUS IN ANIMAL MECHANICS.

The Relation between the Actual and Potential or Inherent Work of Fan-shaped Muscles.—Fan-shaped muscles are muscles whose one attachment (origin or insertion) is a point, and whose other attachment (insertion or origin) is a line. These muscles are divisible into three groups:

(1) **Circular Muscles,** *i.e.* those in which the line of attachment is the arc of a circle (*e.g.* Pectoralis Major, Latissimus Dorsi, etc.).

(2) **Elliptical Muscles,** *i.e.* those in which the line of attachment is a portion of the circumference of an ellipse (*e.g.* Temporalis, etc.).

(3) **Triangular Muscles,** *i.e.* those in which the line of attachment is a straight line (*e.g.* Trapezius).

The circular muscles are the easiest ones to deal with mathematically, whilst the elliptical ones are the most complicated of all. Here we shall deal with the circular and triangular varieties only. But first, we must understand what is meant by the *actual* and *potential work* of a muscle. We have seen (p. 45) that in the case of a prismatic muscle both the total force of the muscular fibres as well as the direction in which the insertion moves towards the origin, are entirely in the line of the fibres themselves, whilst in the case of muscles of the rhomboid class, in which the fibres are not attached to their origin and insertion perpendicularly, the **effective component** of the contractile force of each fibre is less than the actual force with which the fibre contracts, and also the **effective movement** of the insertion is not the same as the actual amount by which the fibres contract. Hence we say that the *actual work* of a muscle is the product of its effective force by the effective shortening. The *potential* or *inherent work* of a muscle is the amount of work that the muscle is capable of doing under the most favourable arrangement of its fibres (*i.e.* if its fibres are arranged in a prismatic manner), when both the force of its contraction as well as the direction of shortening are entirely in the direction of its fibres.

(*a*) **Circular Muscle** (fig. 109).—Let AB = circular arc representing the origin of the muscle, and O = centre of the circle, representing the insertion. Let $\angle$ AOB = 2θ, and let OC be the middle fibre bisecting the angle AOB, so that $\angle$ AOC = $\angle$ BOC = θ. Also let L = length of each fibre. Then, since each fibre must contract with the same force,

$\therefore$ resultant of any pair of fibres like OA and OB, or OD and OE, etc., making equal angles at opposite sides of the bisector OC, will act along OC.

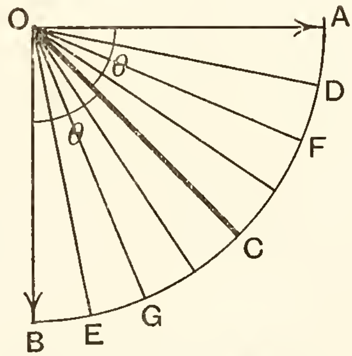


FIG. 109.—Arrangement of Fibres in a Circular Muscle.

$\therefore$ OC will represent the direction of the resultant of the whole muscle.

Now since the component of the force f of, say, the fibre OA along OC $= f\cos\theta$,

$\therefore \dfrac{dR}{d\theta} = f\cos\theta$ (R being the resultant of *all* the fibres),

$$\therefore \quad dR = f\cos\theta\, d\theta = \text{force of a single fibre.}$$

$$\therefore \quad R = \int_{-\theta}^{+\theta} f\cos\theta\, d\theta = 2f\sin\theta,$$

i.e. the resultant R of the forces of all the fibres, estimated in the direction OC, is equal to $2f\sin\theta$ . . . . . (i)

Also, since the length of each fibre is the same = L,

$\therefore$ the amount of shortening of each fibre during contraction is also the same $= l$ (say);

$\therefore$ amount of shortening in direction of resultant $= l$;

$\therefore$ *actual total work of muscle* $= 2fl\sin\theta$ . . (ii)

But potential or inherent work of the muscle is the amount of work that all fibres are **capable** of performing, and as each fibre acts with a force f, and contracts by an amount l,

$\therefore$ amount of inherent work of each fibre $= fl$.

$\therefore$ amount of *inherent work of whole muscle* $= 2fl\theta$;

$$\therefore \quad \frac{\text{actual work}}{\text{inherent work}} \text{ of a circular muscle} = \frac{2fl\sin\theta}{2fl\theta} = \frac{\sin\theta}{\theta}.$$

Example.—Find the amount of work lost by (1) the *Pectoralis Major*, (2) the *Latissimus Dorsi*, (3) the *Iris*, as the result of the circular fan-shaped arrangement of their fibres, given that the angle between the extreme fibres of these muscles is:

(1) In the case of the *Pectoralis Major* = 90°.
(2) In the case of the *Latissimus Dorsi* = 35°.

(3) In the case of the *Iris* = 180° (since the fibres radiate through a semicircle). (See S. Haughton, "*Animal Mechanics.*")

(1) $\frac{\text{Actual work}}{\text{Inherent work}}$ of *Pectoralis Major*

$$= \frac{\sin 45°}{\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{3{\cdot}14}{4}}$$

$$= \frac{2\sqrt{2}}{3{\cdot}14} = 0{\cdot}90.$$

∴ Amount of work lost = 10 per cent.

(2) $\frac{\text{Actual work}}{\text{Inherent work}}$ of *Latissimus Dorsi*

$$= \frac{\sin 17° \; 30'}{0{\cdot}3054} = \frac{0{\cdot}3007}{0{\cdot}3054} = 0{\cdot}98.$$

∴ Amount of work lost = 2 per cent.

(3) In the case of the *Iris*, $\theta = 90$.

$$\therefore \quad \frac{\sin \theta}{\theta} = \frac{1}{\pi/2} = 0{\cdot}64.$$

∴ Amount of work lost = 36 per cent.

(*b*) **Triangular Muscle** (fig. 110).—

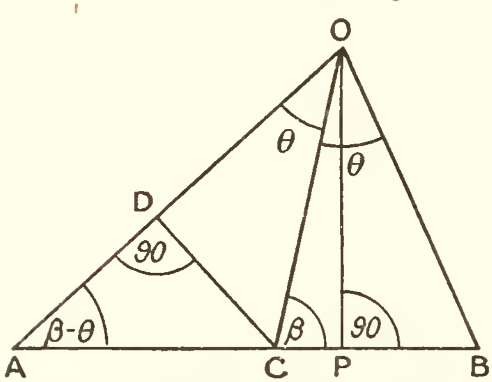


Fig. 110.—Diagram for a Triangular Muscle.

O = Insertion.
AB = Origin.
∠AOB = 2θ.
OC = Bisector of angle AOB making an angle β with the base.
OP = Perpendicular to base.
CD = Perpendicular from C to side OA.

Each fibre contracts with the same force f independent of the length of the fibre (which, of course, is variable).

Let L be the variable length of a fibre and l be the amount of shortening of a fibre, which varies with L

$$\left(\frac{l}{\mathrm{L}} \text{ being constant } = \mathrm{K}\right).$$

Now, since the contractile force of each fibre is the same,

∴ OC represents the direction of the resultant of the whole muscle (as in the case of a circular muscle).

∴ As in the case of a circular muscle, the resultant of the forces of all the fibres estimated in the direction OC is

$$R = \int_{-\theta}^{+\theta} f \cos\theta \, d\theta = 2f \sin\theta.$$

Now let b = amount of shortening of the bisector OC;

∴ actual work done by triangular muscle $= 2fb \sin\theta$.

But $$\frac{b}{OC} = K,$$

$$\therefore \quad b = K . OC;$$

$$\therefore \quad \text{actual work } W = 2fK . OC \sin\theta.$$

But $$\frac{DC}{OC} = \sin\theta,$$

$$\therefore \quad DC = OC \sin\theta.$$

$$\therefore \quad \textit{Actual work } W = 2fK . DC \quad . \quad . \quad . \quad (1)$$

Now let us find the *inherent work* of the muscle.

$$\frac{OP}{OC} = \sin\beta.$$

$$\therefore \quad OP = OC \sin\beta.$$

Also, if we take any fibre such as OA, whose length = L, we have

$$\frac{OP}{OA} \text{ or } \frac{OP}{L} = \sin(\beta - \theta),$$

$$\therefore \quad OP = L \sin(\beta - \theta).$$

$$\therefore \quad L \sin(\beta - \theta) = OC \sin\beta.$$

$$\therefore \quad L = \frac{OC \sin\beta}{\sin(\beta - \theta)}.$$

∴ l (*i.e.* amount of shortening of any fibre) which = KL,

$$= \frac{K . OC \sin\beta}{\sin(\beta - \theta)}.$$

But force of contraction of a fibre = f.

∴ Amount of work inherent in a fibre

$$= \frac{f . K . OC \sin\beta}{\sin(\beta - \theta)}.$$

$\therefore$ *Inherent work* of whole triangular muscle is given by

$$W' = f \, . \, K \, . \, OC \sin \beta \int_{-\theta}^{+\theta} \frac{d\theta}{\sin(\beta - \theta)}$$

$$= f \, . \, K \, . \, OC \sin \beta \, . \log_e \frac{\cot \frac{1}{2}(\beta - \theta)}{\cot \frac{1}{2}(\beta + \theta)} \text{ (see p. 251).}$$

$$= f \, . \, K \, . \, OP \, . \log_e \frac{\cot \frac{1}{2}(\beta - \theta)}{\cot \frac{1}{2}(\beta + \theta)} \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad \frac{\text{Actual work}}{\text{Inherent work}} \left(i.e. \ \frac{W}{W'} \right) \text{ of a triangular muscle}$$

$$= \frac{2fK \, . \, DC}{fK \, . \, OP \, . \log_e \dfrac{\cot \frac{1}{2}(\beta - \theta)}{\cot \frac{1}{2}(\beta + \theta)}} = \frac{2DC}{OP \log_e \dfrac{\cot \frac{1}{2}(\beta - \theta)}{\cot \frac{1}{2}(\beta + \theta)}}.$$

Example.—Find the amount of work lost during the contraction of the two Trapezii muscles, the following measurements being given (fig. 111):—

$\beta = 83°.$
$\theta = 47° \, 30'.$
$OP = 7$ ins.
$DC = 5 \cdot 16$ ins.

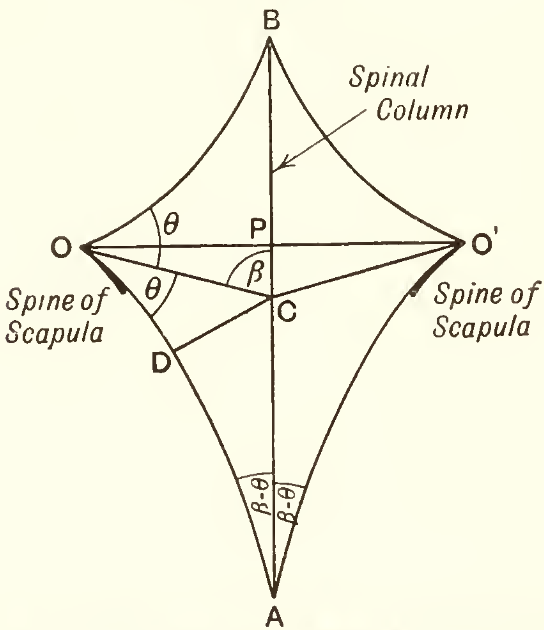


Fig. 111.—Diagram for Calculating the Work done by the Contraction of the two Trapezii Muscles.

$$\frac{W}{W'} = \frac{2DC}{OP \log_e \dfrac{\cot \frac{1}{2}(\beta - \theta)}{\cot \frac{1}{2}(\beta + \theta)}}$$

$$= \frac{2 \times 5{\cdot}16}{7 \log_e \dfrac{\cot \frac{1}{2}(83^\circ - 47^\circ\, 30')}{\cot \frac{1}{2}(83^\circ + 47^\circ\, 30')}}$$

$$= \frac{10{\cdot}32}{7 \times 2{\cdot}303 \log_{10} \left(\dfrac{\cot 17^\circ\, 45'}{\cot 65^\circ\, 15'}\right)} = 0{\cdot}77.$$

$\therefore$ Each Trapezius muscle loses 23 per cent. of its inherent work, in virtue of the fan-shaped arrangement of its fibres.

CHAPTER XVIII.

USE OF THE INTEGRAL CALCULUS FOR DETERMINING LENGTHS, AREAS AND VOLUMES, ALSO CENTRES OF GRAVITY AND MOMENTS OF INERTIA.

Length of a Curve.—Supposing it is required to find the length of the curve APQR in fig. 113:

Since $\quad QR^2 = (\delta x)^2 + (\delta y)^2,$

$$\therefore \quad QR = \sqrt{(\delta x)^2 + (\delta y)^2},$$

i.e. $$\delta l = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \,.\, \delta x \text{ (where } l = \text{length of curve).}$$

But in the limit the chord QR (fig. 114) is equal to the arc QR (fig. 113),

$$\therefore \quad dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx,$$

$$\therefore \quad l = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx.$$

Hence, if the equation of the curve is known, $\frac{dy}{dx}$ is known and l can be calculated between any two limits.

Example.—What is the length of the circumference of a circle of radius r?

Equation of circle is $\quad x^2 + y^2 = r^2$

$$\therefore \quad y^2 = r^2 - x^2$$

$$\therefore \quad 2y\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{y^2}}$$

$$= \frac{1}{y}\sqrt{x^2 + y^2}$$

$$= \frac{r}{y}$$

$$= \frac{r}{\sqrt{r^2 - x^2}}$$

$$\therefore \quad l = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}} \, . \, dx = 4r\left[\sin^{-1}\frac{x}{r}\right]_0^r \text{ (see p. 245)}$$
$$= 2\pi r.$$

Graphical Calculation.—As an example we will find the length of the portion of the curve $y = x^2/2$ between $x = \frac{1}{2}$ and $x = 3$ (fig. 112).

Take OA = AB = BC, etc., along the x axis, each, say, equal

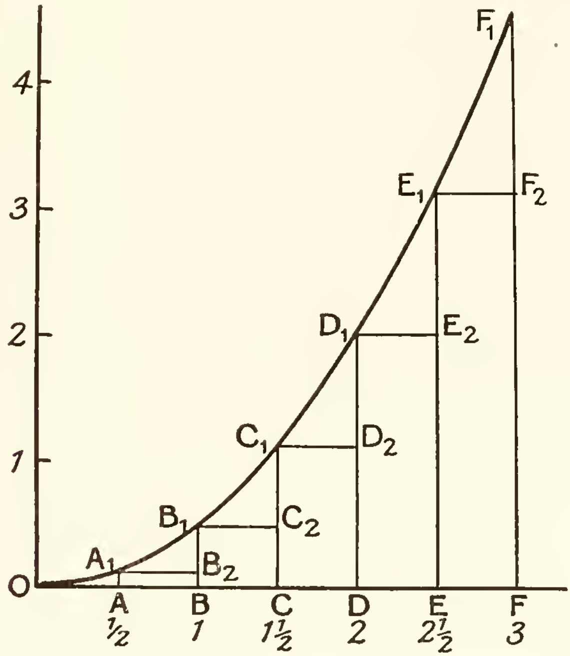


FIG. 112.—Graphical Method of Finding the Length of the Curve $y=\dfrac{x^2}{2}$.

to half a unit ($= \delta x$). Draw the ordinates AA_1, BB_1, CC_1, etc., at each of these points to meet the curve in A_1, B_1, C_1, etc. Draw A_1B_2, B_1C_2, C_1D_2, etc., parallel to the x axis (*i.e.* perpendicular to BB_1, CC_1, etc.). Then the portions of the curve A_1B_1, B_1C_1, etc., being very small may be considered practically as straight lines and as equal to the hypotenuses of the right-angled triangles $A_1B_1B_2$, $B_1C_1C_2$, etc.

$\therefore \quad A_1B_1 = \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{(0{\cdot}5)^2 + (0{\cdot}5 - 0{\cdot}125)^2} = 0{\cdot}625$ (since δy in this case $= BB_1 - AA_1 =$ value of y at point where

$x = 1$ less the value of y at the point where $x = 1/2$, $= \frac{1}{2} - \frac{0{\cdot}5^2}{2}$ $= 0{\cdot}5 - 0{\cdot}125$).

Similarly,

$$
\begin{array}{lll}
B_1C_1 & = \sqrt{(0{\cdot}5)^2 + (1{\cdot}125 - 0{\cdot}5)^2} & = 0{\cdot}800 \\
C_1D_1 & & = 1{\cdot}008 \\
D_1E_1 & & = 1{\cdot}239 \\
\text{and}\quad E_1F_1 & & = 1{\cdot}463
\end{array}
$$

$\therefore$ whole length A_1F_1 is approximately equal to $5{\cdot}135$

By integration we have

$$l = \int_{0\cdot5}^{3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{0\cdot5}^{3} \sqrt{(1 + x^2)}\,dx \quad \left(\text{since } \frac{dy}{dx} = x\right)$$

$$= \left[\frac{x\sqrt{1+x^2}}{2} + \frac{2{\cdot}3}{2}\log_{10}(x + \sqrt{1+x^2})\right]_{0\cdot5}^{3} \quad \text{(p. 246.)}$$

$$= \left\{\frac{3\sqrt{10}}{2} + 1{\cdot}15\log(3 + \sqrt{10})\right\} - \left\{\frac{0{\cdot}5\sqrt{1{\cdot}25}}{2} + 1{\cdot}15\log(0{\cdot}5 + \sqrt{1{\cdot}25})\right\}$$

$$= 5{\cdot}132.$$

Hence the two results differ only by about 0·1 per cent.

Areas.—The finding of areas is of great importance in all practical mathematical work.

Let AB (fig. 113) be any curve whose equation is $y = f(x)$, and

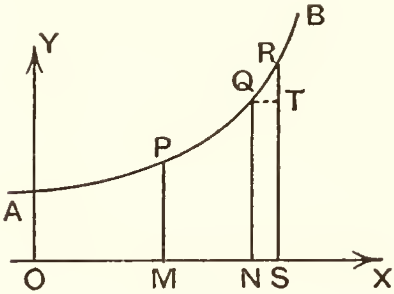


Fig. 113.

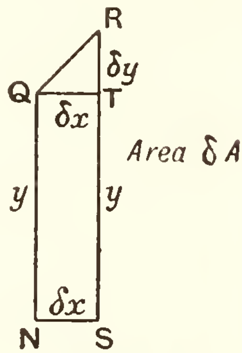


Fig. 114.

let it be required to find the area (A) of the portion PMNQ.

If $\quad ON = x$, then $NQ = y$.

Now take $NS = \delta x$, and draw the ordinate SR,

then $\quad SR = y + \delta y$

and area $PMSR = A + \delta A$.

$$\therefore \text{ Area QNSR, which } = PMSR - PMNQ$$
$$= A + \delta A - A$$
$$= \delta A.$$

Now if the short distance QR were straight, then, as in fig. 114, we would have

$$\text{Area} \qquad \delta A = y \,.\, \delta x + \tfrac{1}{2}\delta x \,.\, \delta y.$$

$$\therefore \qquad \frac{\delta A}{\delta x} = y + \tfrac{1}{2}\delta y.$$

$\therefore$ As δx and δy get smaller and smaller and become dx and dy, δA becomes dA and we ultimately get (in the limit when dx and $dy = 0$)

$$\frac{dA}{dx} = y.$$

$$\therefore \quad dA = ydx = f(x)dx.$$

$$\therefore \quad \int dA = \int ydx = \int f(x)dx.$$

$\therefore$ $A = F(x)$ where $F(x)$ is the integral of $f(x)dx$. Hence, if we have any curve the integral of whose equation is known or can be found, then the area, A, enclosed by it can be determined.

Whenever we have to find the area enclosed by any curve we have always to integrate between limits. Thus, in the foregoing case, the limits of x are OM, ON, because we are concerned with finding the area, *not* of the surface under the whole curve, but only that under the portion PQ, which is limited on the left by PM and on the right by QN.

EXAMPLE.

Find the area of a circle of radius R.

The whole area may be considered as being made up of a series of concentric rings. Consider one such ring. It consists of two concentric circles whose radii are a and $a+da$. As the circumference of the inner of these two circles $= 2\pi a$, and the circumference of the outer of these two circles $= 2\pi(a+da)$,

$\therefore$ Area of ring $= (2\pi a + \pi da)da$ $= 2\pi ada$ [since $(da)^2$ is of the second order of magnitude].

$\therefore$ Area of circle

$$= 2\pi\int_0^R ada = \pi R^2.$$

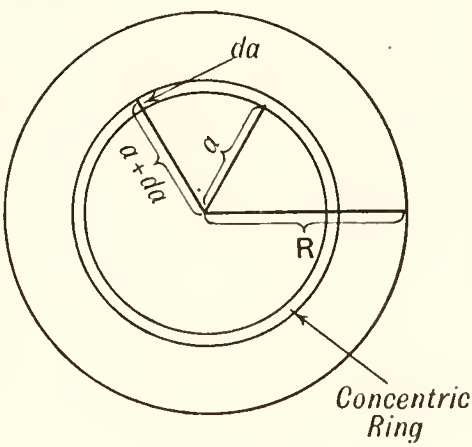


FIG. 115.

Graphical Methods.—Plot the function whose integral is to be determined, and find the area (enclosed by the curve between

the ordinates at two given values of x and the x axis) in one of the following ways:—

(*a*) Count the squares on the squared paper on which the area has been traced (fig. 116).

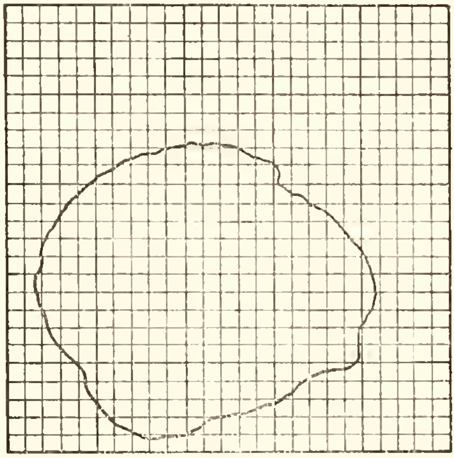


FIG. 116.—Squared Paper Method of Approximate Integration.

(*b*) Cut out the area, weigh it, and compare its weight with that of a known area of the same paper. This is a method adopted for finding the surface area of the human body.

(*c*) *Mean Ordinate Method* (fig. 117).—Since any area can be

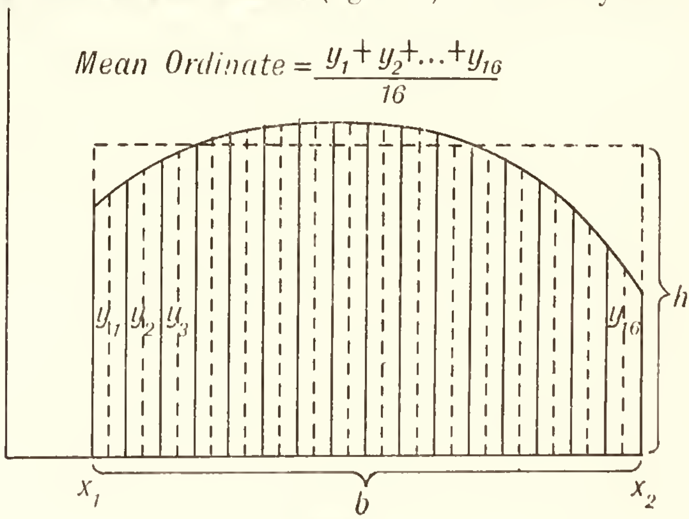


FIG. 117.

split up into a number of trapezia, as in the figure, it is obvious that the whole area is approximately equal to the product of the base and the *average height* of all these trapezia. Hence,

on dividing the base into a number of equal parts and drawing perpendiculars at the mid-point of each (dotted lines in the figure), the area of the figure is obtained by multiplying the base by $\dfrac{\text{(the sum of all the ordinates)}}{\text{number of ordinates}}$

$$= \text{base} \times \text{mean ordinate.}$$
$$= bh \text{ (where } h = \text{height of mean ordinate).}$$

See application of this method in the case of Harmonic Analysis, p. 341.

The Newton-Cotes rule and Simpson's rule can only be mentioned here by name.

The Planimeter.—This is a special instrument that rapidly and accurately measures the area of any irregular closed curve.

The type commonly used is *Amsler's planimeter* (fig. 118). It

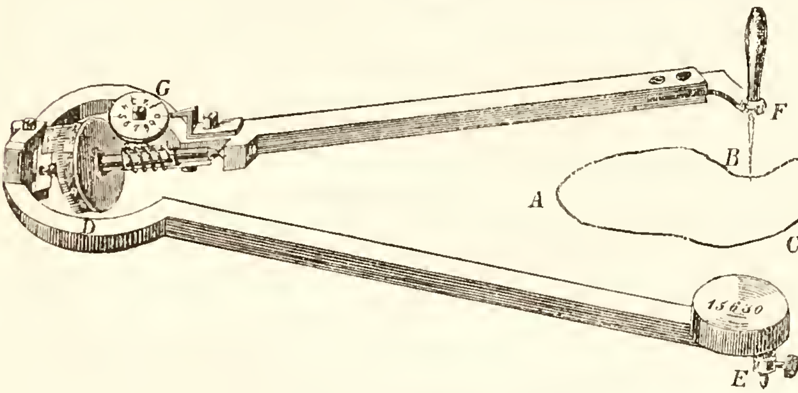


FIG. 118.—Amsler's Planimeter.

consists of two arms pivoted together. The end of one arm (E) is fixed down to any point either inside or outside the area, whilst the point (F) fixed to the end of the other arm is made to move round the outline of the curve. A graduated wheel records the area traced out.

Area of Surface of Revolution.—Imagine the curve AB (fig. 113) to rotate about OX as an axis. Each point on it will trace out the circumference of a circle. Thus the point Q will trace out the circumference of a circle of radius NQ ($= 2\pi \,.\, \text{NQ}$), whilst the point R will trace out the circumference of a circle of radius SR ($= 2\pi \,.\, \text{SR}$). Hence the small arc QR, which we may call δS, will trace out a surface of revolution whose approximate area is equal to $2\pi\left(\dfrac{\text{NQ}+\text{SR}}{2}\right)\delta\text{S}$. Hence in the limit when δS becomes $d\text{S}$, and NQ becomes equal to SR, the

area of the surface of revolution of the curve becomes

$$\int 2\pi \mathrm{NQ} \,.\, d\mathrm{S} = 2\pi\int \mathrm{SR} \,.\, d\mathrm{S}$$

But $\mathrm{NQ} = y$ and $d\mathrm{S} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$

$\therefore$ Area of surface of revolution $= 2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$

Hence if the equation of the curve is known, the area of the surface of revolution can easily be determined.

EXAMPLES.

(1) Find the area of the curved surface of a sphere.

A sphere is formed by the revolution of a circle round its diameter as axis.

The equation of a circle is $y^2+x^2 = r^2$, or $y = \sqrt{r^2-x^2}$.

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{\sqrt{r^2-x^2}} \quad \text{and} \quad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{r}{\sqrt{r^2-x^2}}.$$

$\therefore$ *Area of surface of sphere* $= 2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\pi\int\sqrt{r^2-x^2}\,.\,\frac{r}{\sqrt{r^2-x^2}}\,.\,dx = 2\pi r\int dx = 2\pi rx.$$

Taking the limits of integration as $+r$ and $-r$ we have

$$\text{Area of surface of sphere} = 2\pi r\int_{-r}^{+r} dx = 2\pi r\Big[\,x\,\Big]_{-r}^{+r} = 4\pi r^2.$$

(2) The equation of an ellipse being $\frac{y^2}{b^2}+\frac{x^2}{a^2} = 1$ (see p. 121), find the area of the surface of the prolate spheroid formed by the revolution of the ellipse about its major axis.

$$y = \frac{b}{a}\sqrt{a^2-x^2}, \quad \therefore \quad \frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}$$

$$\text{Required area} = 2\pi\int_{-a}^{+a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\frac{b}{a}\int_{-a}^{+a}\sqrt{a^2-x^2}\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\,dx$$

$$= 2\pi\frac{b}{a}\int_{-a}^{+a}\frac{\sqrt{a^4-x^2(a^2-b^2)}}{a}dx.$$

But if e be the eccentricity of an ellipse

$$(a^2 - b^2) = e^2a^2 \text{ (see p. 121).}$$

$$\therefore \quad \text{Required area} = \frac{2\pi b}{a^2}\int_{-a}^{+a}\sqrt{a^4 - x^2a^2e^2}\,.\,dx$$

$$= \frac{2\pi b}{a}\int_{-a}^{+a}\sqrt{a^2 - x^2e^2}\,.\,dx$$

$$= \frac{2\pi be}{a}\int_{-a}^{+a}\sqrt{\frac{a^2}{e^2} - x^2}\,.\,dx$$

$$= \frac{2\pi be}{a}\left[\frac{x\sqrt{\frac{a^2}{e^2} - x^2}}{2} + \frac{a^2}{2e^2}\sin^{-1}\frac{xe}{a}\right]_{-a}^{+a} \text{ (see p. 246)}$$

$$= \frac{2\pi be}{a}\left[\frac{a^2\sqrt{1 - e^2}}{e} + \frac{a^2}{e^2}\sin^{-1} e\right]$$

$$= 2\pi ba\left(\sqrt{1 - e^2} + \frac{\sin^{-1} e}{e}\right) \text{ (see Example (3), p. 311)}$$

Graphical Calculation of Area of Surface of Revolution.—In cases where $y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$ cannot readily be integrated, the area may be evaluated graphically, with a very considerable degree of accuracy, by summing the areas traced out by successive portions of the curve taken at small but convenient equal intervals of x. The curve is drawn and the lengths of the successive portions of the curve arc found in the manner described on p. 293. If these small portions OA_1, A_1B_1, B_1C_1, etc. are designated δS_1, δS_2, δS_3, etc., then if we take the mid-ordinates y_1, y_2, y_3 . . ., whose lengths are found either from the equation $y = \phi x$, or by inspection on the squared paper in cases where there is no known relationship between y and x, the areas traced out by these portions δS_1, δS_2, etc. of the curve are obviously approximately equal to $2\pi y_1 . \delta S_1$, $2\pi y_2 . \delta S_2$, $2\pi y_3 . \delta S_3$. . . .

$$\therefore \quad \text{Total area} = 2\pi(y_1\delta S_1 + y_2\delta S_2 + y_3\delta S_3 + \ldots).$$

As an illustration we will evaluate the area of the surface of revolution (between $x = 0$ and $x = 3$) of the parabola $y^2 = 4x$ when rotated about its axis and compare the result

obtained by the graphical method with the accurate result found by integration.

In fig. 119 $OA = AB = BC = \ldots = EF = 0{\cdot}5$ unit.

$\therefore$ From $y = 2\sqrt{x}$,

$$AA_1 = \sqrt{2} = 1{\cdot}414 \quad \text{and} \quad OA_1 = \sqrt{2{\cdot}25} = 1{\cdot}500$$

$$BB_1 = \sqrt{4} = 2{\cdot}000 \qquad A_1B_1 = \sqrt{(0{\cdot}5)^2 + (0{\cdot}586)^2} = 0{\cdot}770$$

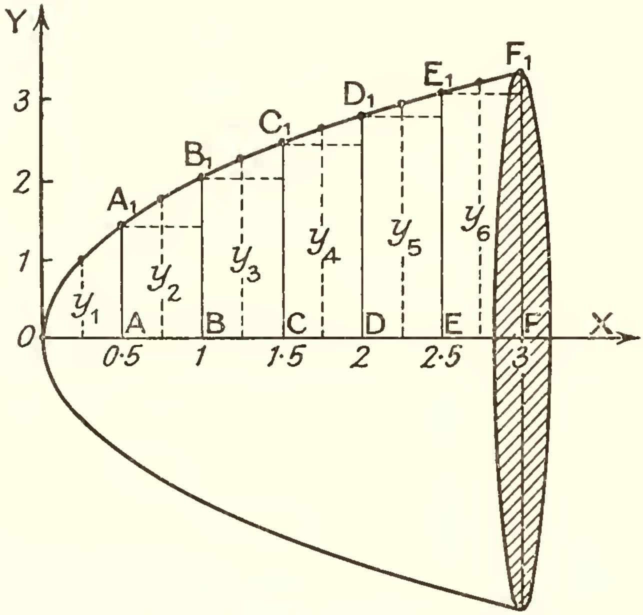


FIG. 119.—Graphical Method of Finding the Area of the Surface of Revolution of $y^2 = 4x$.

Similarly,

$$CC_1 = 2{\cdot}449 \qquad B_1C_1 = 0{\cdot}673$$

$$DD_1 = 2{\cdot}828 \qquad C_1D_1 = 0{\cdot}627$$

$$EE_1 = 3{\cdot}162 \qquad D_1E_1 = 0{\cdot}601$$

$$FF_1 = 3{\cdot}464 \qquad E_1F_1 = 0{\cdot}584.$$

Further, mid-ordinate

$$y_1 = \sqrt{1} = 1{\cdot}000$$

$$y_2 = \sqrt{3} = 1{\cdot}732$$

$$y_3 = 2{\cdot}236$$

$$y_4 = 2{\cdot}646$$

$$y_5 = 3{\cdot}000$$

$$y_6 = 3{\cdot}316.$$

$\therefore$ Portion of surface traced out by

$$\begin{aligned}
OA_1 &= 2\pi y_1 \,.\, OA_1 = 2\pi \times 1 \times 1{\cdot}5 && = 2\pi \times 1{\cdot}500 \\
A_1B_1 &= 2\pi y_2 \,.\, A_1B_1 = 2\pi \times 1{\cdot}732 \times 0{\cdot}77 && = 2\pi \times 1{\cdot}334
\end{aligned}$$

Similarly, portion of surface traced out by

$$\begin{aligned}
B_1 C_1 &= 2\pi \times 1{\cdot}505 \\
C_1 D_1 &= 2\pi \times 1{\cdot}659 \\
D_1 E_1 &= 2\pi \times 1{\cdot}803 \\
E_1 F_1 &= 2\pi \times 1{\cdot}937
\end{aligned}$$

$\therefore$ *Total surface* traced out by curve $OF_1 = 2\pi \times 9{\cdot}738$
$= 61{\cdot}19$

By integration we have as follows:—

Total area of paraboloid

$$\begin{aligned}
&= 2\pi\int_0^3 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx \\
&= 2\pi\int_0^3 2\sqrt{x}\sqrt{1+\frac{1}{x}}\,.\,dx \\
&= 4\pi\int_0^3 (1+x)^{\frac{1}{2}}dx \\
&= 4\pi\int_0^3 (1+x)^{\frac{1}{2}}d(x+1)
\end{aligned}$$

[Since $d(x+1) = dx$]

$$\begin{aligned}
&= 4\pi\left[\tfrac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3 \\
&= \frac{8\pi}{3}\left[(4)^{\frac{3}{2}} - 1\right] = 58{\cdot}67.
\end{aligned}$$

Thus the graphical result exceeds the correct value by 4 per cent. Had the intervals in the former method been made shorter, the result would have been much more accurate; but the working would of course have been even much more laborious. The student will notice that integration not only gives exact results, but is accompanied by no tedious arithmetic and occupies practically no time.

Volumes.—If the curve in fig. 113 revolves about the x axis it will produce a volume of revolution. As the area of the section produced by the ordinate QN is $\pi QN^2 = \pi y^2$, therefore the total volume is $\pi\int y^2dx$. Thus in the case of a sphere, which of course is the volume of revolution of a circle about its diameter, we have

$$\text{Volume of sphere} = \pi\int_{-r}^{+r} y^2dx$$
$$= \pi\int_{-r}^{+r}(r^2 - x^2)dx \text{ (where } r = \text{radius)}$$
$$= 2\pi\int_0^r (r^2 - x^2)dx$$
$$= 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r$$
$$= 2\pi\left(r^3 - \frac{r^3}{3}\right)$$
$$= \frac{4\pi r^3}{3}.$$

EXAMPLE.

Find the volume of the prolate spheroid formed by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x (or major) axis.

$$\text{Volume} = 2\pi\int_0^a y^2dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)dx$$
$$= 2\pi\frac{b^2}{a^2}\left[\left(a^2x - \frac{x^3}{3}\right)\right]_0^a$$
$$= \tfrac{4}{3}\pi b^2a.$$

An oblate spheroid is generated by the revolution of an ellipse about the y (or minor) axis, and its volume $= \frac{4}{3}\pi a^2b$.

Centres of Gravity.—It is proved in textbooks on mechanics that if a system of particles $m_1, m_2, m_3, m_4, \ldots m_n$ be placed at the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), \ldots (x_n, y_n)$, respectively, in the plane of the co-ordinate axes, the position of the centre of gravity of the system $(\bar{x}, \bar{y})$ is given by the formulæ:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + \ldots + m_nx_n}{m_1 + m_2 + m_3 + m_4 + \ldots + m_n},$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 + \ldots + m_ny_n}{m_1 + m_2 + m_3 + m_4 + \ldots + m_n};$$

or $$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \text{and} \quad \bar{y} = \frac{\Sigma my}{\Sigma m}.$$

Hence in the case of a continuous body,

$$\bar{x} = \frac{\int x\,dm}{\int dm}, \quad \bar{y} = \frac{\int y\,dm}{\int dm}, \quad \text{where} \quad \int dm = \text{total mass of body}.$$

The formulæ $\bar{x} = \frac{\Sigma mx}{\Sigma m}$, $\bar{y} = \frac{\Sigma my}{\Sigma m}$, enable one to calculate the position of the centre of gravity of an irregular figure of uniform density, such as a section of bone, by graphical methods:

An outline of the figure is drawn on squared paper and two convenient lines OX, OY are taken as the axes (fig. 120). Lines

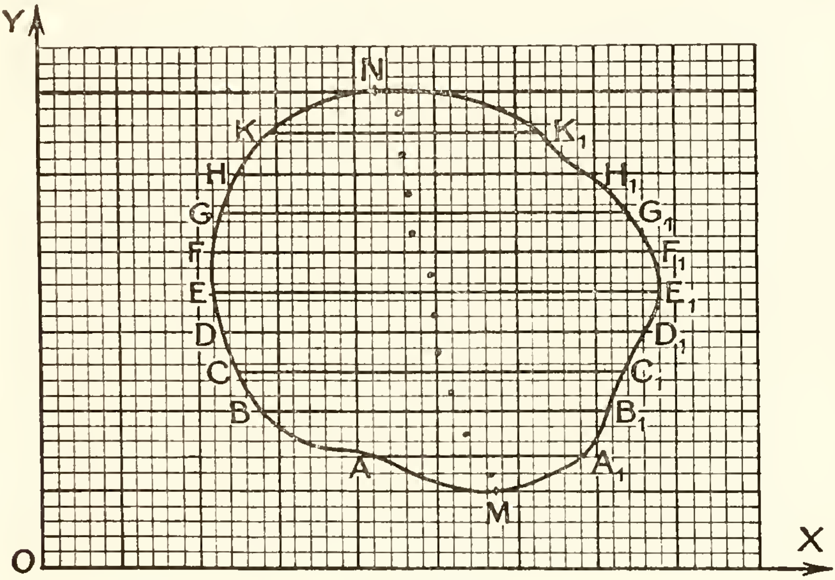


FIG. 120.

AA_1, BB_1, . . . KK_1 are drawn parallel to the X axis. The area of each separate section AMA_1, ABB_1A_1, BCC_1B_1, etc., is estimated by counting the number of separate squares in each. These numbers will respectively represent the separate masses of each of the sections. Now mark with a dot the position (as estimated by the eye) of the centre of gravity of each separate section and note the distances of each from OY and OX. We then have the values of m_1, m_2, m_3, . . . m_n, as well as of (x_1, y_1), (x_2, y_2), (x_3, y_3), . . . (x_n, y_n), and hence $\bar{x} = \frac{\Sigma mx}{\Sigma m}$ and $\bar{y} = \frac{\Sigma my}{\Sigma m}$ can readily be computed.

Determination of Moments of Inertia.—In the study of locomotion, as well as of the physics of bone and in the construction of various physiological recording apparatus, the moment of inertia is a constant of great importance. The integral calculus affords a ready method of evaluating this constant.

Definitions.—(1) The moment of inertia of a *particle* about a

given point or line is the product of the mass of that particle by the square of its distance from the point or line.

(2) The moment of inertia of a *body* about a given point or line is the sum of the products of the masses of every particle in the body by the square of its own distance from the point or line.

If m be the mass of unit volume of the substance,
dx be the size of the particle,
and x be the distance of the particle from the point or line,
then $m\int x^2dx = \mathrm{I}$ (where I stands for the moment of inertia).

(*Note.*—Kinetic energy $= \frac{1}{2}\mathrm{I}\omega^2$ where ω = angular velocity.)

EXAMPLES.

(1) To find the moment of inertia of a circle about its centre (*i.e.* about a line passing as an axis through its centre).

Let ABC (fig. 121) represent the circle (of radius r) and A′B′C′ a thin annulus of thickness dx at a distance x from the centre.

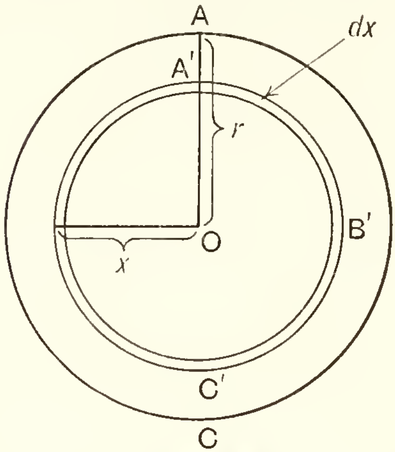


Fig. 121.

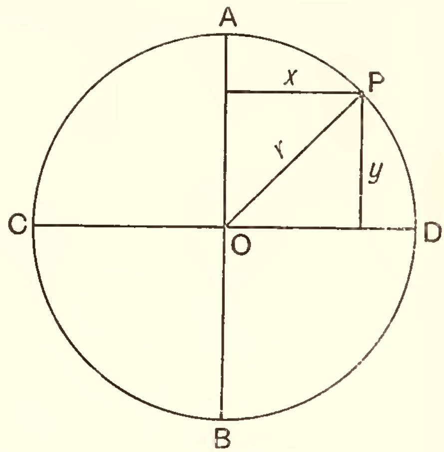


Fig. 122.

Then the moment of inertia of every particle in the annulus about O is
mx^2dx (m = mass of unit volume of substance);

∴ Moment of inertia of entire annulus about O is

$$2\pi x \,.\, mx^2dx = 2\pi mx^3dx,$$

∴ Moment of inertia of whole circle is given by

$$\mathrm{I} = 2\pi m\int_0^r x^3dx = \frac{\pi mr^4}{2}.$$

(2) Find the moment of inertia of a circle about one of its diameters (*e.g.* a transverse section of a circular bone).

The moment of inertia of any particle P (fig. 122) about the diameter AB is $\mathrm{P}x^2$ (where P represents the mass).

The moment of inertia of P about the diameter CD is similarly Py^2.

∴ Sum of these moments of inertia

$$= Px^2 + Py^2 = P(x^2 + y^2)$$
$$= Pr^2 = \text{moment of inertia of P about O,}$$

i.e. the moment of inertia of the particle about O is equal to the sum of the moments of inertia of the particle about the two diameters.

∴ The moment of inertia of the whole circle about the centre is equal to the sum of the moments of inertia of the whole circle about its diameters at right angles.

But the moment of inertia of a circle about one diameter is equal to that about the other diameter,

∴ Moment of inertia of circle about a diameter

$$= \tfrac{1}{2} \text{ moment of inertia about centre}$$
$$= \frac{\pi m r^4}{4}.$$

Graphical Method of finding Moment of Inertia.—In the case of irregular bodies like sections of bone, the moment of inertia can readily be calculated graphically in a manner similar to that used for finding the centre of gravity. Thus, in fig. 123,

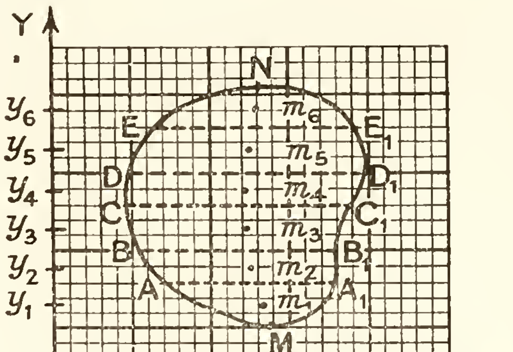


FIG. 123.

the mass m of each of the sections AMA_1, ABB_1A_1, etc., is given by the number of squares it covers, and the situation of each separate centre of gravity is estimated by the eye and its Y ordinate recorded on OY.

Then moment of inertia $= m_1y_1^2 + m_2y_2^2 + \ . \ . \ . + m_6y_6^2$
$= \Sigma my^2.$

This method has been used by J. C. Koch in the case of bone (see *Am. J. Anat.*, 1917).

CHAPTER XIX.

SPECIAL METHODS OF INTEGRATION.

THE integrals so far considered are the so-called standard or fundamental integrals, with which the student is expected to be familiar. When he meets with other expressions which he has to integrate he must by some means bring them into one or other of these standard forms to enable him to perform the integrations. The following are the methods most frequently employed for the purpose:—

(1) **The Method of Substitution,** by which a new variable is introduced which enables the integrand to be put in a simple form. The method will be best understood from a couple of examples.

(1) Find the integral of $\cos^3 x dx$.

Put $\sin x = z$.

$$\therefore \quad \frac{dz}{dx} = \cos x,$$

$$\therefore \quad dx = \frac{dz}{\cos x}.$$

$$\therefore \quad \int \cos^3 x dx = \int \cos^3 x \frac{dz}{\cos x}$$

$$= \int \cos^2 x dz.$$

But $\cos^2 x = 1 - \sin^2 x = 1 - z^2$.

$$\therefore \quad \int \cos^2 x dz = \int (1 - z^2) dz$$

$$= \int dz - \int z^2 dz$$

$$= z - \tfrac{1}{3}z^3 + C$$

$$= \sin x(1 - \tfrac{1}{3}\sin^2 x) + C.$$

(2) Find the value of $\int \frac{dx}{a - \sqrt{x}}$.

Put $x = z^2$, when dx becomes $2zdz$, and $\sqrt{x}$ becomes z.

$$\therefore \quad \int \frac{dx}{a - \sqrt{x}} = 2\int \frac{z}{a - z}dz.$$

But

$$\frac{z}{a-z} = \frac{a}{a-z} - 1 \text{ (identically)}$$

$$\therefore \quad 2\int \frac{z}{a-z}dz = 2\int \frac{a}{a-z}dz - 2\int dz$$

$$= 2[-a \log_e (a-z) - z] + \mathrm{C}$$

$$= 2\left[a \log_e \frac{1}{a-z} - z\right] + \mathrm{C}$$

i.e.

$$\int \frac{dx}{a - \sqrt{x}} = 2\left[a \log_e \frac{1}{a - \sqrt{x}} - \sqrt{x}\right] + \mathrm{C}$$

$$= 2\left[2{\cdot}3a \log_{10} \frac{1}{a - \sqrt{x}} - \sqrt{x}\right] + \mathrm{C} \text{ (see p. 336).}$$

(2) **The Method of Partial Fractions.**—By this method a complicated algebraic fraction is broken up into a sum of a number of simpler fractions which render themselves very suitable for integration.

Example.—Find the value of $\int \frac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6}dx.$

Splitting the expression up into its partial fractions in the manner described on p. 30, one finds that

$$\frac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6} = \frac{9}{x-2} - \frac{11}{2(x-3)} - \frac{5}{2(x-1)}.$$

$$\therefore \quad \int \frac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6}dx = 9\int \frac{1}{x-2}dx - \tfrac{11}{2}\int \frac{1}{x-3}dx - \tfrac{5}{2}\int \frac{1}{x-1}dx$$

$$= 9 \log_e (x-2) - \tfrac{11}{2} \log_e (x-3) - \tfrac{5}{2} \log_e (x-1) + \mathrm{C}$$

$$= \log_e \frac{(x-2)^9}{\sqrt{(x-3)^{11}(x-1)^5}} + \mathrm{C}.$$

EXERCISE.

Find the value of $\int \frac{dx}{(x+1)(x+2)^2(x^2+1)}$

[*Answer*, $\tfrac{1}{2} \log_e(x+1) + \tfrac{1}{5}\frac{1}{(x+2)} - \tfrac{9}{25} \log_e(x+1) - \tfrac{7}{5} \log_e \sqrt{x^2+1} - \tfrac{1}{50} \tan^{-1} x.$]

(3) **The Method of Trigonometrical Transformation,** whereby trigonometrical differential expressions are reduced to easily integrable forms.

EXAMPLES.

(1) Find the value of $\int \cos^2 \theta d\theta$.

Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1$,

$$\therefore \quad \cos^2 \theta = \frac{\cos 2\theta + 1}{2}.$$

$$\therefore \quad \int \cos^2 \theta d\theta = \tfrac{1}{2}\int (\cos 2\theta + 1) d\theta$$

$$= \tfrac{1}{2}\int \cos 2\theta d\theta + \tfrac{1}{2}\int d\theta$$

$$= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.$$

(2) An important integral is $\int \sqrt{a^2 - x^2}\, dx$.

It can easily be integrated by transforming it into the form $\int \cos^2 \theta d\theta$, thus causing the square root to disappear, as follows:—

Put $x = a \sin \theta$, then $\sqrt{a^2 - x^2}$ becomes $a \cos \theta$ and dx becomes $a \cos \theta d\theta$.

$$\therefore \quad \int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2 \theta d\theta$$

$$= a^2\left(\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right) + C$$

$$= \frac{a^2}{2}(\sin \theta \cos \theta + \theta) + C$$

$$= \frac{a^2}{2}\left(\frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + \sin^{-1}\frac{x}{a}\right) + C$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C \text{ (see p. 246).}$$

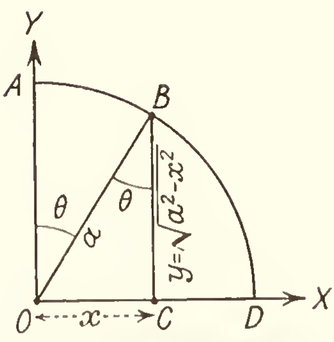


FIG. 124.—Geometrical Evaluation of $\int \sqrt{a^2 - x^2} dx$.

Fig. 124 verifies this result pictorially. ADO is a quadrant of a circle of radius a. $\int \sqrt{a^2 - x^2} dx$ is obviously the area OCBA = triangle OBC $\left(= \tfrac{1}{2}x\sqrt{a^2 - x^2}\right)$ + sector OBA $\left(= \frac{a^2}{2}\theta \text{ (see p. 54)} = \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)$.

$$\therefore \quad \int \sqrt{a^2 - x^2} dx = \tfrac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

In general, whenever one has to deal with algebraic expressions involving quadratic surds, one makes some

trigonometrical substitution which will get rid of the square root sign. Thus:

(3)
$$\int \frac{dx}{\sqrt{x^2 - a^2}}.$$

Put $x = a \sec \theta$ and we have

$$\sqrt{x^2 - a^2} = a\sqrt{\sec^2 \theta - 1} = a \tan \theta,$$

and dx becomes $a \sec \theta \tan \theta d\theta$.

$$\therefore \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta} = \int \sec \theta d\theta.$$

But $\sec \theta$ is the differential coefficient of $\log_e (\sec \theta + \tan \theta)$ (see Example (8), p. 192).

$$\therefore \quad \int \sec \theta d\theta = \log_e (\sec \theta + \tan \theta) + C$$

$$= \log_e \left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right) + C \text{ (see p. 245).}$$

(4) **Integration by Parts.**—This method is based on the rule for differentiating a product of two functions.

Thus
$$d(uv) = vdu + udv.$$
$$\therefore \quad vdu = d(uv) - udv.$$
$$\therefore \quad \int vdu = \int d(uv) - \int udv.$$
$$\therefore \quad \int vdu = uv - \int udv.$$

Hence, if the integral of vdu is not known, but that of udv is known, then by means of this formula, $\int vdu$ can be found.

Let us take again $\int \cos^2 \theta d\theta$ considered on p. 306.

Put
$$\cos \theta d\theta = dv \quad \text{and} \quad \cos \theta = u,$$

then
$$\int \cos^2 \theta d\theta = \int udv = uv - \int vdu$$
$$= \cos \theta \,.\, \sin \theta + \int \sin^2 \theta d\theta$$

(since $v = \int dv = \int \cos \theta d\theta = \sin \theta$ and $du = - \sin \theta d\theta$)

$$= \tfrac{1}{2} \sin 2\theta + \int (1 - \cos^2 \theta) d\theta,$$

i.e.
$$\int \cos^2 \theta d\theta = \tfrac{1}{2} \sin 2\theta + \theta - \int \cos^2 \theta d\theta;$$

$$\therefore \qquad 2\int \cos^2 \theta d\theta = \tfrac{1}{2} \sin 2\theta + \theta.$$

$$\therefore \qquad \int \cos^2 \theta d\theta = \tfrac{1}{4} \sin 2\theta + \frac{\theta}{2} + C.$$

Example.—Find $\int \log x dx$.

Let $\qquad u = \log x, \quad$ and $\quad dv = dx.$

$$\therefore \quad \int \log x dx = \int u dv = uv - \int v du$$

$$= x \log x - \int x \cdot \frac{dx}{x}$$

$$= x \log x - x + C.$$

Multiple Integration.—Just as it is possible to differentiate a function successively, so it is possible to perform successive integration. Thus, if

$$\frac{d^3y}{dx^3} = a, \quad \text{then} \quad \frac{d^2y}{dx^2} = ax + C_1.$$

A second integration gives $\qquad \dfrac{dy}{dx} = \tfrac{1}{2}ax^2 + C_1x + C_2.$

A third and final integration yields $y = \tfrac{1}{6}ax^3 + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$

These operations might have been written as follows:—

$$\int \frac{d^3y}{dx^3} \cdot dx = ax + C_1.$$

$$\int\int \frac{d^3y}{dx^3} \cdot dx \cdot dx = \tfrac{1}{2}ax^2 + C_1x + C_2.$$

$$\int\int\int \frac{d^3y}{dx^3} \cdot dx \cdot dx \cdot dx = \tfrac{1}{6}ax^3 + \tfrac{1}{2}C_1x^2 + C_2x + C_3.$$

In this case the successive integrations have been performed with respect to the same independent variable. It is sometimes necessary, however, to perform the task with respect to a different independent variable each time. Thus we might have a multiple integration as follows:—

$$\int\int\int U \, dx \, dy \, dz,$$

where U stands for some function of x, y, z. The expression would then mean that we have to integrate first with respect to one of the variables (say x), treating the other variables as if they were constants, then integrate the result with respect to any other of the variables (say y), again treating the other variables as if they were constants, and finally integrate the second result with respect to the third variable (z), the others being considered constants.

Thus if $\int U\,dx = A$, then $\iiint U\,dx\,dy\,dz = \iint A\,dy\,dz$,

and if $\int A\,dy = B$, then $\iint A\,dy\,dz = \int B\,dz$.

Each integration can be performed between given limits,

e.g. $$\int_0^a\int_0^b\int_0^c U\,dx\,dy\,dz.$$

EXAMPLES.

(1) Evaluate $\int_{-\infty}^{+\infty} e^{-x^2}dx$.

Since $$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy \text{ (p. 253)} = \text{say, I},$$

$$\therefore \quad \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \mathrm{I}^2.$$

Put $y = xv$, when dy becomes xdv.

$$\therefore \quad \int_0^\infty e^{-y^2}dy = \int_0^\infty xe^{-x^2v^2}dv,$$

$$\therefore \qquad \mathrm{I}^2 = \int_0^\infty e^{-x^2}dx\int_0^\infty xe^{-x^2v^2}dv = \int_0^\infty\int_0^\infty xe^{-x^2(1+v^2)}dxdv.$$

But $\int_0^\infty xe^{-x^2(1+v^2)}dx = \dfrac{1}{2(1+v^2)}$ (p. 254),

$$\therefore \quad \mathrm{I}^2 = \int_0^\infty\int_0^\infty xe^{-x^2(1+v^2)}dxdv = \int_0^\infty \frac{dv}{2(1+v^2)} = \frac{\pi}{4} \text{ (p. 254)},$$

$$\therefore \quad \mathrm{I} = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

$$\therefore \quad \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}.$$

(2) Using the value of $\int_{-\infty}^{+\infty} e^{-x^2}dx$, find the value of $\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}dx$.

Put $\dfrac{x^2}{2\sigma^2} = z^2$, so that $x = (\sigma\sqrt{2})z$, and $dx = (\sigma\sqrt{2})dz$.

$$\therefore \quad \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}dx = \sigma\sqrt{2}\int_{-\infty}^{+\infty} e^{-z^2}dz = \sigma\sqrt{2}\int_{-\infty}^{+\infty} e^{-x^2}dx = \sigma\sqrt{2\pi}.$$

This is a most important integral in statistical theory (see p. 408).

(3) The gravid uterus may be considered as a prolate spheroid, *i.e.* the solid of revolution generated by the rotation of an ellipse about its major

axis. Find its volume and its surface if its major and minor axes are 12 and 8 ins. respectively. Also find the radius of a sphere having the same volume as the uterus.

Since the volume of an ellipsoid $= \frac{4}{3}\pi b^2 a$ (see p. 302),

$\therefore$ in the case of the uterus, where $a = 6$ ins. and $b = 4$ ins.,

$$\text{volume} = \tfrac{4}{3} \times \tfrac{22}{7} \times 16 \times 6 = 402{\cdot}3 \text{ cub. ins.}$$

The surface of a prolate spheroid is as we have seen (p. 299)

$$= 2\pi ab\left(\frac{\sin^{-1} e}{e} + \sqrt{1-e^2}\right),$$

$\left(\text{where } e \text{ is the } \textit{eccentricity} \text{ (p. 121)} = \sqrt{1 - \dfrac{b^2}{a^2}}\right).$

$$= 2\pi \,.\, 6 \,.\, 4\left(\frac{\sin^{-1}\sqrt{\frac{20}{36}}}{\sqrt{\frac{20}{36}}} + \tfrac{4}{6}\right)$$

$$= 48\pi\left(\frac{\sin^{-1}\frac{1}{3}\sqrt{5}}{\frac{1}{3}\sqrt{5}} + \tfrac{2}{3}\right)$$

$$= 48\pi\left(\frac{0{\cdot}84 \text{ radian}}{0{\cdot}745} + 0{\cdot}67\right)$$

$$= 271{\cdot}3 \text{ sq. ins.}$$

If a and b are equal, then the spheroid becomes a sphere, and its volume $\frac{4}{3}\pi ab^2$ becomes $\frac{4}{3}\pi a^3 = \frac{4}{3}\pi r^3$ (where r = radius).

$$\therefore \quad \tfrac{4}{3}\pi r^3 = \tfrac{4}{3}\pi \,.\, 16 \,.\, 6.$$

$$\therefore \quad r^3 = 96.$$

$$\therefore \quad 3 \log r = \log 96 = 1{\cdot}98227.$$

$$\therefore \quad r = 4{\cdot}579.$$

$\therefore$ Mean curvature of uterus is that of a sphere whose radius $= 4{\cdot}579$ ins.

Approximate Integration.—There are some functions the integral of which cannot be expressed in finite terms. For example, $\int e^{x^2}dx$ cannot be expressed in finite terms because we do not know any function whose differential coefficient is e^{x^2}. Another integral of the same type is $\int e^{-x^2}dx$. Such integrals, however, are frequently met with in practical work. Thus $\int e^{-x^2}dx$ is of great importance in statistical work because the equation of the probability curve—whose area between given limits of x has to be found—is $y = Ae^{-\frac{x^2}{2\sigma^2}}$ (see p. 408). In physics also this particular integral is frequently encountered. There are several methods by which such integrals may be calculated.

Method of Converting into an Infinite Series.—

EXAMPLES.

(1) Find the value of $\int e^{-x^2}dx.$

Expanding e^{-x^2} into an infinite series we get

$$e^{-x^2} = 1 - \frac{x^2}{1} + \frac{x^4}{1\,.\,2} - \frac{x^6}{1\,.\,2\,.\,3} + \frac{x^8}{1\,.\,2\,.\,3\,.\,4} - \ldots + \ldots$$

$$\therefore \quad \int e^{-x^2}dx = \int dx - \int x^2dx + \tfrac{1}{2}\int x^4dx - \tfrac{1}{6}\int x^6dx + \tfrac{1}{24}\int x^8dx - \ldots + \ldots$$

$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 - \tfrac{1}{42}x^7 + \tfrac{1}{216}x^9 - \ldots + \ldots + C.$$

(2) $$\int_0^{\frac{1}{2}} e^{-x^2}dx = \tfrac{1}{2} - \tfrac{1}{24} + \tfrac{1}{320} - \tfrac{1}{5376} + \ldots = 0{\cdot}461.$$

For the value of the integral when x is equal to ∞ see p. 311.

(3) Find the value of x which makes

$$\frac{1}{\sqrt{\pi}}\int e^{-x^2}dx = \tfrac{1}{4}.$$

Here we have to find x from the equation

$$\int e^{-x^2}dx = \frac{\sqrt{\pi}}{4} = 0{\cdot}4431.$$

Expanding and integrating as in the first example we get

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \ldots + \ldots = 0{\cdot}4431.$$

Ignoring in this case terms of x higher than x^5, we may put

$$y = x - \frac{x^3}{3} + \frac{x^5}{10} - 0{\cdot}4431.$$

We therefore have to find a value of x which will make $y = 0$ (see p. 127). Tabulating the values of y corresponding to various values of x, we get the following table:—

x	x	$-\frac{x^3}{3}$	$+\frac{x^5}{10}$	$-0{\cdot}4431$	y
0	0	0	0	−0·4431	−0·4431
0·1	0·1	−0·0003	+0·000001	−0·4431	−0·3434
0·2	0·2	−0·0026	+0·000032	−0·4431	−0·2457
0·3	0·3	−0·0090	+0·00024	−0·4431	−0·1519
0·4	0·4	−0·0213	+0·00102	−0·4431	−0·0634
0·5	0·5	−0·0417	+0·00313	−0·4431	+0·0183

Plot the curve, which will be found to cut the x axis at the point where x is between 0·4 and 0·5, as can be seen from the table, since when $x = 0{\cdot}4$, y is −ve, and when $x = 0{\cdot}5$, y is +ve. Hence x must lie between 0·4 and 0·5.

Now tabulate the corresponding values of y and x for $x = 0{\cdot}41, 0{\cdot}42, 0{\cdot}43, \ldots 0{\cdot}49$, when it will be found that x lies between 0·47 and 0·48.

We may get a still closer approximation by tabulating for values of $x = 0{\cdot}471, 0{\cdot}472, \ldots$ etc., when x is found to lie between 0·476 and 0·477.

Tabulating still further we get $x = 0{\cdot}4769$ to four places of decimals.

(4) Find the value of π.

We know that
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \mathrm{C} \quad . \quad . \quad . \quad . \qquad \text{(A)}$$

But
$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}\cdot\frac{x^2}{1} + \frac{3}{4}\cdot\frac{x^4}{2!} + \frac{15}{8}\cdot\frac{x^6}{3!} + \ldots$$
$$\therefore \int \frac{dx}{\sqrt{1-x^2}} = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \tfrac{15}{336}x^7 + \ldots + \mathrm{C}. \quad . \quad . \qquad \text{(B)}$$

Equating A and B we get
$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots$$

Now, if $\qquad x = \frac{1}{2}$ then $\sin^{-1} x = \dfrac{\pi}{6}$.
$$\therefore \quad \frac{\pi}{6} = \tfrac{1}{2} + \tfrac{1}{6}\cdot\tfrac{1}{8} + \tfrac{3}{40}\cdot\tfrac{1}{32} + \tfrac{5}{112}\cdot\tfrac{1}{128} + \ldots + \mathrm{C},$$
$$= 0{\cdot}52351 + \mathrm{C}.$$

But since $\sin^{-1} 0 = 0, \quad \therefore \quad \mathrm{C} = 0.$
$$\therefore \quad \frac{\pi}{6} = 0{\cdot}52351, \text{ whence } \pi = 3{\cdot}141 \ldots$$

EXERCISES.

(1) Prove that $\log_e (1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$
$$\left[\int \frac{dx}{1+x} = \log_e (1+x) \text{ and also } = \int (1 - x + x^2 - x^3 + \ldots)dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

(2) Find a value of x which satisfies the equation
$$x^2 - 5 \log_{10} x - 2{\cdot}531 = 0.$$

Proceeding as in Example (3), p. 313, we have as follows:—

x	1·99	2·00	2·01	2·02
y	−0·054	−0·036	−0·007	+0·022

Hence x lies between 2·01 and 2·02. By tabulating further (and plotting if desired), x is found to be 2·012.

(3) Solve the equation $x + \log x = 2$. [*Answer*, $x = 1{\cdot}755$.]

(4) In the theory of developmental mechanics the following equation occurs:—

$$\frac{\pi}{4} - \theta + \theta \cot^2 \theta - \cot \theta = 0.$$ (See "*Child Physiology*," p. 101.)

Find the value of θ which will satisfy this equation (notice that in the equation θ is in radians).

Plotting $y = \frac{\pi}{4} - \theta$ and $y = \theta \cot^2 \theta - \cot \theta$ gives a rough value of θ.

Tabulating we get:

θ (in degrees)	$\frac{\pi}{4}$	$-\theta$	$-\cot \theta$	$+\theta \cot^2 \theta$	y
34° 37′	0·7854	−0·6042	−1·4487	+1·2680	0·0005
34° 38′	0·7854	−0·6045	−1·4478	+1·2671	0·0002
34° 39′	0·7854	−0·6048	−1·4469	+1·2661	−0·0002

$\therefore$ θ must lie between 34° 38′ and 34° 39′. θ = 34° 38′ 30″.

Graphical and **Instrumental** methods of integration have already been considered. (See Chapter XVIII.)

CHAPTER XX.

DIFFERENTIAL EQUATIONS.

Definitions.—(1) A differential equation is an equation which connects the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc., with the variables x and y themselves.

E.g.
$$\frac{dy}{dx} = \text{K}.$$
$$\frac{dy}{dx} + ay - b = 0.$$
$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 9y = 0.$$
$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$$

Differential equations are of great importance in all scientific work, since not only can many of the so-called "laws" be expressed in their most general form by such equations, but the equations continually occur as the result of the mathematical analysis of various phenomena.

(2) By the *solution* of a differential equation is meant the obtaining from such an equation of another called the *primitive*, which contains the variables alone, without the differentials. Thus, as we shall see presently, the solution of

$$b\frac{dy}{dx} + ay = 0,$$

is $\qquad y = \text{A}e^{-\frac{a}{b}x}$, where A is a constant.

(3) **Order and Degree of a Differential Equation.**—The *order* of a differential equation is determined by the order of the highest differential coefficient occurring in it. Thus, an equation containing $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ . . . or $\frac{d^ny}{dx^n}$, is called an equation of the 1st, 2nd, 3rd . . . or nth order. The *degree* of a differ-

ential equation is determined by the degree or highest power of the differential coefficient occurring in it. Thus, an equation containing $\frac{dy}{dx}$, $\left(\frac{dy}{dx}\right)^2$, $\left(\frac{dy}{dx}\right)^3$ . . . or $\left(\frac{dy}{dx}\right)^n$ is called an equation of the 1st, 2nd, 3rd . . . or nth degree.

Examples.

$$3\frac{d^2y}{dx^2} + 25\frac{dy}{dx} - 18y = 0$$

is an equation of the second order and first degree, *i.e.* a linear equation of the second order; but

$$7\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

is an equation of the first order and second degree.

As the solution of a differential equation essentially entails a number of integrations (varying with the order of the equation), and as each successive integration introduces a new integration constant (p. 238), it follows that the **complete** or **general** solution of a differential equation of the nth order will contain n new arbitrary constants, C_1, C_2, C_3, . . . C_n. Thus, as we saw on p. 310, if $\frac{d^3y}{dx^3} = a$, then $y = \frac{1}{6}ax^3 + \frac{1}{2}C_1x^2 + C_2x + C_3$.

The student will realise that *if more than one value of y will solve a differential equation, then the complete or general solution will be the algebraic sum of these values.*

Thus, since in the case of either $y = C_1 \sin nx$, or $y = C_2 \cos nx$, we have $\frac{d^2y}{dx^2} = -n^2y$ (p. 190), therefore the solution of $\frac{d^2y}{dx^2} + n^2y = 0$ is not only $y = C_1 \sin nx$, or $y = C_2 \cos nx$, but obviously also $y = C_1 \sin nx + C_2 \cos nx$. Indeed, the last is the **general** or **complete** solution of the differential equation, and the other two are **particular** solutions obtained from the general solution by taking one or other of the constants as zero.

Similarly, since in the case of either $y = C_1e^{nx}$, or $y = C_2e^{-nx}$ we have $\frac{d^2y}{dx^2} = n^2y$, therefore the **complete** solution of $\frac{d^2y}{dx^2} - n^2y = 0$ is $y = C_1e^{nx} + C_2e^{-nx}$, of which $y = C_1e^{nx}$ and $y = C_2e^{-nx}$ are **particular** solutions.

Further, since in either of the four cases $y = C_1 \sin nx$, $y = C_2 \cos nx$, $y = C_3 e^{nx}$ and $y = C_4 e^{-nx}$, we have $\frac{d^4y}{dx^4} = n^4 y$, therefore the general or complete solution of $\frac{d^4y}{dx^4} - n^4y = 0$ is $y = C_1 \sin nx + C_2 \cos nx + C_3 e^{nx} + C_4 e^{-nx}$, and each of the other four solutions, as well as the sum of any two or three of them, is a particular solution in which the other three, two or one of the arbitrary constants are taken as zero.

(4) **Homogeneous and Non-homogeneous Equations.**—A differential equation is homogeneous or non-homogeneous according as the sum of the exponents of the variables is or is not the same in each term of the equation.

Thus
$$(ax + by)dx + (a'x + b'y)dy = 0$$
and
$$x^2 + 2xy\frac{dy}{dx} - y^2 = 0$$
are homogeneous,

but
$$(ax + by + c)dx + (a'x + b'y + c')dy = 0$$
and
$$(x + y^2)dx + 2xydy = 0$$
are non-homogeneous.

(5) **Exact and Non-exact Equations.**—A differential equation is exact or inexact according as it is presented exactly as derived by the differentiation of a function of x and y, or it has subsequently been modified by cancelling out some common factor consisting of some function of x and y.

Thus $(x + y^2)dx + 2xydy = 0$ is an exact differential equation because it has been derived directly from the differentiation of $\frac{1}{2}x^2 + xy^2 = c$.

On the other hand, $ydx - xdy = 0$ is a non-exact equation because after differentiation of the primitive $\left(\text{viz. } \frac{x}{y} = c\right)$ yielding $\frac{1}{y^2}(ydx - xdy) = 0$, the factor $\frac{1}{y^2}$ has been cancelled out. When this factor is restored, the equation, $\frac{1}{y^2}(ydx - xdy) = 0$, becomes exact.

Integrating Factor.—While every exact equation can be solved directly, a non-exact equation can only be solved after it has been made exact by restoring the factor which has been cancelled

out. Such a factor is, therefore, called an *integrating factor* and is generally denoted by the letter μ.

The great difficulty of solving non-exact equations consists in finding the appropriate integrating factor—although it is to be noted that not only has every non-exact equation **some** integrating factor, but it has in theory, at any rate, an infinite number of such factors.

Thus, let $ydx - xdy = 0$.

We have seen that $\frac{1}{y^2}$ is an integrating factor yielding the primitive $\frac{x}{y} = c$.

Similarly $\frac{1}{x^2}$ is an integrating factor yielding the primitive $\frac{y}{x} = c_1$.

Again, $\frac{1}{xy}$ is also an integrating factor, converting the equation into $\frac{dx}{x} - \frac{dy}{y} = 0$, whose solution is $\log_e \frac{x}{y} = c_2$.

Euler's Criterion of Integrability.—If the equation be expressed in the most general form $\mathrm{M}dx + \mathrm{N}dy = 0$, where M and N are functions of x and y, then the equation is exact if $\frac{\delta \mathrm{M}}{\delta y} = \frac{\delta \mathrm{N}}{\delta x}$.

Thus $(x + y^2)dx + 2xydy = 0$ is exact, because

$$\frac{\delta(x+y^2)}{\delta y} = 2y \quad \text{and} \quad \frac{\delta(2xy)}{\delta x} = 2y,$$

so that
$$\frac{\delta \mathrm{M}}{\delta y} = \frac{\delta \mathrm{N}}{\delta x}.$$

On the other hand, $ydx - xdy = 0$ is not exact, because

$$\frac{\delta y}{\delta y} = +1 \quad \text{and} \quad \frac{\delta(-x)}{\delta x} = -1,$$

so that
$$\frac{\delta \mathrm{M}}{\delta y} \text{ is not equal to } \frac{\delta \mathrm{N}}{\delta x}.$$

By restoring any one of the integrating factors, however, the equation becomes exact.

Thus, putting $\mu = \frac{1}{y^2}$, the equation becomes $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$, and

$$\frac{\delta M}{\delta y} = -\frac{1}{y^2}; \quad \text{also} \quad \frac{\delta N}{\delta x} = \frac{\delta\left(-\frac{x}{y^2}\right)}{\delta x} = -\frac{1}{y^2}; \quad \therefore \quad \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

Putting $\mu = \frac{1}{x^2}$, the equation becomes $\frac{ydx}{x^2} - \frac{1}{x}dy = 0$, and both $\frac{\delta M}{\delta y}$ and $\frac{\delta N}{\delta x} = \frac{1}{x^2}$.

Similarly, $\mu = \frac{1}{xy}$ makes the equation $\frac{dx}{x} - \frac{dy}{y} = 0$, and both $\frac{\delta M}{\delta y}$ and $\frac{\delta N}{\delta x}$ are equal to 0.

For the methods of finding integrating factors the student is referred to the standard textbooks. In this chapter only a few of the simpler types of differential equations can be dealt with.

Solution of Differential Equations.

Separation of Variables.—The first and most essential point in the solution of differential equations is to "separate the variables," so as to group all the x's with the (dx)'s and all the y's with the (dy)'s. When the resulting equation is integrated the solution is obtained, giving an equation containing the variables alone without the differentials. Two cases arise:—

(1) **Variables Directly Separable.**—Such equations are generally easily soluble.

EXAMPLES.

(1) $ydx + xdy = 0$.

We separate the variables as follows:—
Divide throughout by xy and get

$$\frac{dx}{x} + \frac{dy}{y} = 0.$$

$$\therefore \quad \int\frac{dx}{x} + \int\frac{dy}{y} = C,$$

$$i.e. \quad \log_e x + \log_e y = \log_e A$$

(where $\log_e A = C$, the constant being put in the logarithmic form to make it uniform with $\log_e x$ and $\log_e y$),

$$\text{or} \quad \log_e xy = \log_e A.$$

$$\therefore \quad xy = A \text{ is the solution.}$$

(2) Similarly the solution of $ydx - xdy = 0$

is
$$\frac{x}{y} = \text{A}.$$

(3) Solve the equation
$$\frac{dy}{dx} = \sqrt{1 - y^2}.$$

Separating the variables we get
$$\frac{dy}{\sqrt{1 - y^2}} = dx.$$
$$\therefore \quad \int \frac{dy}{\sqrt{1 - y^2}} = \int dx + \text{C},$$

or
$$\sin^{-1} y = x + \text{C} \text{ (see p. 190)}.$$
$$\therefore \quad y = \sin (x + \text{C}).$$

(4) Solve
$$ay + b\frac{dy}{dx} = 0.$$

Separation of variables gives
$$\frac{b}{a}\frac{dy}{y} = -dx.$$
$$\therefore \quad \frac{b}{a}\int \frac{dy}{y} = -\int dx + \text{C},$$

or
$$\frac{b}{a}\log_e y = -x + \text{C}.$$
$$\therefore \quad y = e^{-\frac{ax}{b} + \frac{\text{C}a}{b}} = e^{-\frac{ax}{b}} . e^{\frac{\text{C}a}{b}},$$
$$\therefore \quad y = \text{A}e^{-\frac{a}{b}x}, \text{ where A} = e^{\frac{a\text{C}}{b}}.$$

(5) Solve
$$\frac{dy}{dx} = y + 3$$
$$\int \frac{dy}{y+3} = \int dx + \text{C}.$$
$$\therefore \quad \log_e (y+3) = x + \text{C},$$
$$\therefore \quad y = \text{A}e^x - 3, \text{ where A} = e^{\text{C}}.$$

(6) If
$$\frac{dy}{dx} = n\frac{y}{x}, \text{ then } \frac{dy}{y} = n\frac{dx}{x}.$$
$$\therefore \quad \log_e y = n \log_e x + \log_e \text{A},$$
$$\therefore \quad y = \text{A}x^n.$$

(ii) **Cases where Variables are Not Directly Separable.**—If it is impossible to separate the variables directly it may be possible to effect the separation by introducing a new variable, called an *auxiliary variable.*

(*a*) *Homogeneous Equations.*—In these equations the introduction of a new variable z, such that $z = \frac{y}{x}$ (or $y = xz$) enables the separation to be effected.

Example.—Solve $$x^2 + 2xy\frac{dy}{dx} - y^2 = 0.$$

This is the same as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\frac{y}{x}} \quad . \quad . \quad . \quad \text{(i)}$$

Putting $z = \frac{y}{x}$, or $y = zx$, we have from (i)

$$\frac{d(zx)}{dx} = \frac{z^2 - 1}{2z},$$

i.e.
$$z + \frac{xdz}{dx} = \frac{z^2 - 1}{2z},$$

or
$$\frac{z^2 - 1}{2z} - z = \frac{xdz}{dx},$$

i.e.
$$-\frac{(z^2 + 1)}{2z} = \frac{xdz}{dx}.$$

It is now possible to bring the z's and dz's on one side and the x's and dx's on the other. We then have

$$\int \frac{2zdz}{1 + z^2} = -\int \frac{dx}{x},$$

or $$\log_e(1 + z^2) = -\log_e x + \mathrm{C} = -\log_e x + \log_e \mathrm{A} \text{ (where } \log_e \mathrm{A} = \mathrm{C}),$$

i.e. $$\log_e \frac{(x^2 + y^2)}{x^2} = -\log_e x + \log_e \mathrm{A},$$

or $$\log_e(x^2 + y^2) = \log_e x + \log_e \mathrm{A} = \log_e \mathrm{A}x.$$

$$\therefore \quad x^2 + y^2 = \mathrm{A}x, \text{ whence } y = \sqrt{\mathrm{A}x - x^2}.$$

(*b*) *Non-homogeneous Equations.*—Such equations can generally be rendered homogeneous by the following artifice of getting rid of the term which renders the equation non-homogeneous:—

In the equation $(ax + by + c)dx + (a'x + b'y + c')dy = 0$ put $x = (v + h)$, $y = (w + k)$—where v and w stand for variables and h and k represent constants—when it becomes $\{(av + bw) + (ah + bk + c)\}dv + \{(a'v + b'w) + (a'h + b'k + c')\}dw = 0$.

If now h and k are so chosen (as can be done) that

$$ah + bk + c = 0$$

and
$$a'h + b'k + c' = 0,$$

then the equation reduces to

$$(av+bw)dv+(a'v+b'w)dw = 0,$$

which is homogeneous in v and w and may be solved in the usual manner.

The following example will illustrate the process:—

Solve $$(2x-3y+4)dx+(3x-2y+1)dy = 0.$$

Putting $x = (v+h)$ and $y = (w+k)$, it becomes

$$\{(2v-3w)+(2h-3k+4)\}dv+\{(3v-2w)+(3h-2k+1)\}dw = 0.$$

Now if we choose h and k so that

$$2h-3k+4 = 0$$

and $$3h-2k+1 = 0,$$

i.e. if we make $h = 1$ and $k = 2$ (as found by solving the two simultaneous equations either algebraically or graphically), we get $(2v-3w)dv+(3v-2w)dw = 0$.

This equation being now homogeneous in v and w,

put $z = \dfrac{w}{v}$ or $w = vz$ (as on p. 322) and we have

$$(2-3z)dv+(3-2z)(zdv+vdz) = 0,$$

or $$2(1-z^2)dv-v(2z-3)dz = 0,$$

i.e. $$2\frac{dv}{v} = \frac{(2z-3)}{1-z^2}dz = -\frac{5dz}{2(1+z)}-\frac{dz}{2(1-z)} \text{ (see p. 30);}$$

$$\therefore \quad 4\int\frac{dv}{v} = -5\int\frac{dz}{1+z}-\int\frac{dz}{1-z},$$

or $$4\log_e v = -5\log_e(1+z)+\log_e(1-z)+\text{C}.$$

But $$v = (x-1), \quad w = (y-2) \quad \text{and} \quad z = \frac{w}{v} = \frac{y-2}{x-1},$$

$$\therefore \quad 4\log_e(x-1) = -5\log_e\frac{(x+y-3)}{(x-1)}+\log_e\frac{(x-y+1)}{(x-1)}+\text{C},$$

or $\log_e(x-y+1)-5\log_e(x+y-3)+\log_e\text{A}$ (where $\log_e\text{A} = \text{C}$) $=0$;

$\therefore$ Final solution is

$$\text{A}(x-y+1) = (x+y-3)^5.$$

The method just described fails in the case where $\dfrac{a}{a'} = \dfrac{b}{b'}$. Thus if $(2x-3y+4)dx+(4x-6y+1)dy = 0$, it is impossible to choose h and k so that

$$2h-3k+4 = 0$$

and $$4h-6k+1 = 0.$$

For the proper substitution in such a case the reader is referred to the regular textbooks on differential equations.

General Solution of Linear Equations of the First Order.—The most general type of such an equation is

$$\frac{dy}{dx} + \mathrm{P}y + \mathrm{Q} = 0.$$

The solution which, as we have seen, will contain *one* new constant, is effected by means of the familiar device which gets rid of the term Py as follows:—

Put $y = z\phi(x)$, where z is a new variable and $\phi(x)$ is some arbitrary function of x.

Then, equation becomes

$$\frac{d(z\phi(x))}{dx} + \mathrm{P}z\phi(x) + \mathrm{Q} = 0 \quad . \quad . \quad (1)$$

i.e.

$$\frac{zd\phi(x)}{dx} + \frac{\phi(x)dz}{dx} + \mathrm{P}z\phi(x) + \mathrm{Q} = 0 \text{ (see p. 173)},$$

or

$$\phi(x)\frac{dz}{dx} + z\left\{\frac{d\phi(x)}{dx} + \mathrm{P}\phi(x)\right\} + \mathrm{Q} = 0 \quad . \quad . \quad (2)$$

Now as $\phi(x)$ is an arbitrary function of x we can choose it so as to make the bracketed expression, viz. $\frac{d\phi(x)}{dx} + \mathrm{P}\phi(x)$, vanish (when we get rid of the term containing z).

Putting therefore $\quad \frac{d\phi(x)}{dx} = -\mathrm{P}\phi(x),$

we get $\quad \frac{d\phi(x)}{\phi x} = -\mathrm{P}dx.$

$$\therefore \quad \int\frac{d\phi(x)}{\phi x} = -\int \mathrm{P}dx,$$

giving $\quad \log_e \phi(x) = -\int \mathrm{P}dx$ (see p. 187),

whence $\quad \phi(x) = e^{-\int \mathrm{P}dx}$

Now, since this value of $\phi(x)$ makes the bracketed expression in (2) vanish, therefore equation (2) now becomes

$$\phi(x)\frac{dz}{dx} + \mathrm{Q} = 0, \text{ which is easily solved.}$$

Thus, $\quad dz = -\frac{\mathrm{Q}dx}{e^{-\int \mathrm{P}dx}} = -e^{\int \mathrm{P}dx}\mathrm{Q}dx,$

$$\therefore \quad z = -\int e^{\int P dx} Q dx + C.$$

$$\therefore \quad y = z\phi(x) = e^{-\int P dx}\left[C - \int e^{\int P dx} Q dx\right] \quad . \quad \text{(A)}$$

(C is the integration constant.)

Corollary.—If P and Q, instead of being functions of x, are constants, we obtain

$$y = e^{-Px}\left[C - Q\int e^{Px} dx\right] = e^{-Px}\left[C - \frac{Q}{P}e^{Px}\right] = Ce^{-Px} - \frac{Q}{P} \quad \text{(B)}$$

EXAMPLES.

(1) Solve $\qquad \dfrac{dy}{dx} + \dfrac{y}{x} - x^2 = 0.$

Here $\qquad P = \dfrac{1}{x}; \quad Q = -x^2,$

$$\therefore \quad \int P dx = \int \frac{dx}{x} = \log_e x,$$

$$\therefore \quad e^{\int P dx} = e^{\log_e x} = x,$$

and $\qquad e^{-\int P dx} = e^{-\log_e x} = \dfrac{1}{x},$

$$\therefore \qquad y = \frac{1}{x}\left[C - \int x \,.\, (-x^2) dx\right]$$

$$= \frac{1}{x}\left[C + \int x^3 dx\right]$$

$$= \frac{C}{x} + \tfrac{1}{4}x^4.$$

(2) If L (henries) is the coefficient of self-induction, R (ohms) the resistance, E (volts) the electromotive force, and i (amperes) the current in a circuit, then Ohm's law states that $E = L\dfrac{di}{dt} + Ri$. Find i in terms of R, L and t.

The equation may be written $\dfrac{di}{dt} + \dfrac{R}{L}i - \dfrac{E}{L} = 0.$

As R and L are constants, formula (B) above applies (where $y = i, x = t$, $P = \dfrac{R}{L}$ and $Q = -\dfrac{E}{L}$).

$$\therefore \quad i = Ce^{-\frac{R}{L}t} + \frac{E}{L}\Big/\frac{R}{L} = Ce^{-\frac{R}{L}t} + \frac{E}{R}.$$

Solution of Linear Equations of the Second Order.—As a type we shall take $\frac{d^2y}{dx^2}+\frac{Pdy}{dx}+Qy = 0$, where P and Q are constants.

The complete solution will, as we have seen, contain *two* new arbitrary constants.

Since
$$\frac{d(Ae^{Kx})}{dx} = AKe^{Kx}$$

$$\therefore \quad \frac{d^2y}{dx^2} = AK^2e^{Kx}$$

$\therefore$ If we put $y = Ae^{Kx}$, our equation becomes

$$AK^2e^{Kx}+PAKe^{Kx}+QAe^{Kx} = 0,$$

or
$$Ae^{Kx}(K^2+PK+Q) = 0,$$

$\therefore$ $K^2+PK+Q = 0$. (This equation is called the *auxiliary* equation.)

$$\therefore \quad K = \frac{-P \pm \sqrt{P^2-4Q}}{2} \text{ (p. 20).}$$

The following four possibilities arise, viz.:

(1) $P^2 = 4Q$, when the two roots K_1 and K_2 are each equal to $-\frac{P}{2}$.

Therefore both $y = C_1e^{-\frac{Px}{2}}$ and $y = C_2e^{-\frac{Px}{2}}$ are *particular* solutions satisfying the equation. Nevertheless $y = (C_1+C_2)e^{-\frac{Px}{2}}$ cannot be the *general* solution, because C_1 and C_2 being constants, their sum is another single constant, which we may call C. Hence $y = (C_1+C_2)e^{-\frac{Px}{2}}$ is of the same form as each of the other particular solutions—containing only one term with one arbitrary constant instead of the requisite two terms with two arbitrary constants. To obtain the general solution, therefore, of an equation of the 2nd order in which the two roots of the auxiliary equation are equal, one resorts to the following artifice:—

Put $K_2 = K_1+h$, where h is an indefinitely small quantity that ultimately can be made equal to zero, when K_2 will become equal to K_1. We then have

$$\begin{aligned} y &= C_1e^{K_1x} + C_2e^{(K_1+h)x} \\ &= e^{K_1x}(C_1 + C_2e^{hx}) \\ &= e^{K_1x}\left[C_1 + C_2\left(1 + \frac{hx}{1} + \frac{h^2x^2}{1\,.\,2} + \frac{h^3x^3}{1\,.\,2\,.\,3} + \,.\,.\,.\right)\right] \\ &= e^{K_1x}\left[(C_1 + C_2) + C_2hx\left(1 + \frac{hx}{1\,.\,2} + \frac{h^2x^2}{1\,.\,2\,.\,3} + \,.\,.\,.\right)\right] \end{aligned}$$

As C_2 is an *arbitrary* constant it can be made so large that C_2h is a *finite* constant equal to, say, B. Similarly C_1 can be taken so large and of opposite sign to C_2 that $(C_1 + C_2)$ is another *finite* constant equal to, say, A. Therefore the equation becomes

$$y = e^{K_1x}\left[A + Bx\left(1 + \frac{hx}{1\,.\,2} + \frac{h^2x^2}{1\,.\,2\,.\,3} + \,.\,.\,.\right)\right].$$

But, in the limit, hx and the terms containing higher powers of h vanish.

$\therefore$ Complete solution is $y = (A + Bx)e^{-K_1x} = (A + Bx)e^{-\frac{Px}{2}}$

For verification of this solution, see Example (1), p. 231.

Example. $\dfrac{d^2y}{dx^2} + \dfrac{2dy}{dx} + y = 0.$

Here $K = -1.$

$\therefore$ $y = (A + Bx)e^{-x}$

(2) $P^2 > 4Q$, when there are two roots which are real but unequal, say K_1 and K_2.

$\therefore$ $y = Ae^{K_1x}$ will satisfy the equation.

Also $y = Be^{K_2x}$ will satisfy the equation.

$\therefore$ Complete solution is $y = Ae^{K_1x} + Be^{K_2x}$

Example. $\dfrac{d^2y}{dx^2} + \dfrac{2dy}{dx} - 3y = 0.$

Here $K_1 = 1$ and $K_2 = -3,$

$\therefore$ $y = Ae^{x} + Be^{-3x}$

(3) $P^2 < 4Q$, when the roots are imaginary, viz.:

$$K_1 = \frac{-P + i\sqrt{4Q - P^2}}{2}$$

$$K_2 = \frac{-P - i\sqrt{4Q - P^2}}{2} \text{ (p. 21).}$$

Hence, putting $\sqrt{4Q - P^2} = a$,

$$K_1 = \frac{-P + ia}{2}, \quad K_2 = \frac{-P - ia}{2}.$$

∴ Complete solution is

$$y = Ae^{-\frac{(P-ia)x}{2}} + Be^{-\frac{(P+ia)x}{2}}$$

$$= e^{-\frac{Px}{2}}\left(Ae^{\frac{iax}{2}} + Be^{-\frac{iax}{2}}\right).$$

But $$e^{\frac{iax}{2}} = \cos\frac{ax}{2} + i\sin\frac{ax}{2}$$

and $$e^{-\frac{iax}{2}} = \cos\frac{ax}{2} - i\sin\frac{ax}{2}$$ (see p. 230).

∴ Final solution is

$$y = e^{-\frac{Px}{2}}\left[A\left(\cos\frac{ax}{2} + i\sin\frac{ax}{2}\right) + B\left(\cos\frac{ax}{2} - i\sin\frac{ax}{2}\right)\right]$$

$$= e^{-\frac{Px}{2}}\left[(A+B)\cos\frac{ax}{2} + (A-B)i\sin\frac{ax}{2}\right]$$

or $$y = e^{-\frac{Px}{2}}\left(C_1\cos\frac{ax}{2} + iC_2\sin\frac{ax}{2}\right)$$

where $C_1 = (A+B)$ and $C_2 = (A-B)$.

Example. $$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 3y = 0.$$

Here $K_1 = (-1 + i\sqrt{2})$ and $K_2 = (-1 - i\sqrt{2})$.

∴ $y = e^{-x}(C_1 \cos\sqrt{2}.x + iC_2\sin\sqrt{2}.x)$.

(4) $P = 0$. Then $y = C_1\cos\sqrt{Q}.x + iC_2\sin\sqrt{Q}.x$.

Free and Damped Vibration.—The free vibration of a tuning-fork or a pendulum, and the damped vibration of the mercury in a sphygmomanometer, etc. furnish examples of differential equations of the second order.

(1) *Free Vibration.*—In fig. 125 HO is a tuning-fork (held in a vice at H), of which the free end, P, is plucked aside to D and left to vibrate. Owing to the small distance, on either side of O, through which the free end vibrates, the path of P, viz. DOD′, may be regarded as practically a straight line.

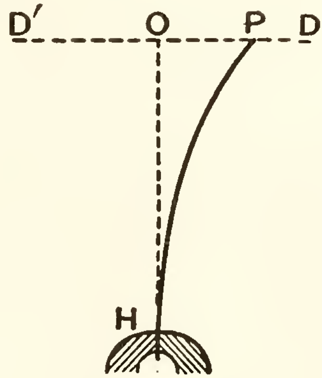


FIG. 125.

Let OP $= x$ and OD($=$ D′O), the extreme limits of vibration, be a. Then experiment shows that the retardation $-\frac{d^2x}{dt^2}$ of the point P at the moment t (as it moves away from O) is proportional to x, *i.e.* $-\frac{d^2x}{dt^2} = n^2x$ (where n^2 is a positive number taken in the square form for convenience, *i.e.* in order to avoid a root sign later).

Therefore $$\frac{d^2x}{dt^2}+n^2x = 0$$

is the differential equation (of the second order) for free vibration, and its solution, as we have seen, is $x = C_1 \sin nt + C_2 \cos nt$ (p. 317).

Our problem now is *to evaluate the constants* C_1 and C_2. This is easily done as follows:—

When, at the beginning of the experiment, the free end of the tuning-fork is at D, then $t = 0$, $x =$ OD $= a$, and velocity, $\frac{dx}{dt}$, $= 0$.

Therefore, putting $x = a$ and $t = 0$ in the equation, we have

$$a = C_1 \sin 0 + C_2 \cos 0 = C_2.$$

$\therefore$ The value of the constant C_2 is a.

Also, differentiating the equation and putting $t = 0$, we get

$$\frac{dx}{dt} = C_1 n \cos 0 - C_2 n \sin 0 = 0.$$
$$\therefore \quad C_1 = 0.$$

Therefore our equation becomes $x = a \cos nt$.

As $\cos nt$ has the same value as $\cos (nt+2\pi)$,

$\therefore$ Our equation finally becomes

$$x = a \cos (nt + 2\pi)$$
$$= a \cos n\left(t + \frac{2\pi}{n}\right)$$

or, what is the same thing,

$$x = a \sin \left(nt + \frac{\pi}{2}\right).$$

a is called the amplitude of the vibration and determines the *loudness* of the note emitted.

Note.—Since $x = a \cos n\left(t + \frac{2\pi}{n}\right)$, it follows that whatever be the position occupied by P at any moment t, that same position will be occupied by it at the moment $\left(t + \frac{2\pi}{n}\right)$. Hence the time of a complete vibration, or the **period** of the vibration—upon which depends the *pitch* of a note—is $2\pi/n$. The period is thus seen to depend upon n and not upon the amplitude a.

This is an example of so-called free or undamped vibration, since there is no friction to retard it.

Corollary.—If $\frac{d^2x}{dt^2} + n^2x = b$, then by putting $b = n^2k$ we obtain

$$\frac{d^2x}{dt^2} + n^2(x - k) = 0, \text{ whence } (x - k) = a \cos n\left(t + \frac{2\pi}{n}\right).$$

(2) *Damped Vibration.*—The oscillation of the mercury inside a sphygmomanometer up and down with the pulse beat is analogous to the vibration of the tuning-fork except that the amplitude of the oscillation is diminished by the friction of the mercury against the wall of the manometer tube.

This retardation is equal to $-u\frac{dx}{dt}$ (where u = coefficient of friction and $\frac{dx}{dt}$ is the velocity of motion).

$$\therefore \quad -\frac{d^2x}{dt^2} - u\frac{dx}{dt} = n^2x \quad \text{or} \quad \frac{d^2x}{dt^2} + u\frac{dx}{dt} + n^2x = 0,$$

which is a general equation of the type $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and represents a damped vibration, *i.e.* a vibration in which there is a damping effect tending to diminish the amplitude of the swing.

Linear Equations of the nth Order.—Equations of the type

$$\frac{d^n y}{dx^n} + \frac{Pd^{n-1}y}{dx^{n-1}} + \ . \ . \ . + \frac{Rdy}{dx} + Sy = 0$$

can be solved in the same way as equations of the second order by putting $y = Ae^{Kx}$, when we get the auxiliary equation

$K^n + PK^{n-1} + \ . \ . \ . + RK + S = 0$ (see Examples below).

Graphical Solution of Differential Equations.—In the same way as it is possible in the case of ordinary equations to find approximate solutions by graphic methods when ordinary algebraic methods fail us (see p. 127), so in the case of differential equations it is often possible to obtain approximate solutions graphically. It would, however, be beyond the scope of this book to enter into a consideration of this subject.

EXAMPLES.

(1) Solve the differential equation:

$$\frac{d^3y}{dx^3} - \frac{6d^2y}{dx^2} + \frac{11dy}{dx} - 6y = 0.$$

The auxiliary equation here is:

$$K^3 - 6K^2 + 11K - 6 = 0,$$

i.e. $$(K-1)(K-2)(K-3) = 0.$$

$\therefore$ Complete solution is $y = Ae^x + Be^{2x} + Ce^{3x}$.

(2) Solve $$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{2dy}{dx} = 0.$$

The auxiliary equation here is $K^3 + K^2 - 2K = 0$,

or $$K(K-1)(K+2) = 0.$$

$\therefore$ Complete solution is $y = A + Be^x + Ce^{-2x}$.

(3) Solve $$\frac{d^2y}{dx^2} + \frac{7dy}{dx} + 10y = 5.$$

Writing the equation as $\dfrac{d^2y}{dx^2} + \dfrac{7dy}{dx} + 10(y - 0{\cdot}5) = 0$,

we can put $$(y - 0{\cdot}5) = Ae^{Kx}.$$

The auxiliary equation is $K^2 + 7K + 10 = 0$, *i.e.* $K_1 = -2$, $K_2 = -5$.

$\therefore$ Complete solution is $y - 0{\cdot}5 = Ae^{-2x} + Be^{-5x}$.

(4) Prove that in the case of ferments—like emulsin—which gradually become destroyed in the course of the reaction which they catalyse, a limit must be reached for the amount of substance decomposed by the ferment, so that even if the original amount of ferment is large and the reaction is allowed to proceed indefinitely, a certain amount of the substance undergoing decomposition under the influence of the ferment must remain unaltered.

Let A = original amount of ferment,
B = original amount of substance which is catalysed,
x = amount of ferment which becomes inactive,
t = time in which this amount x has been destroyed,
y = amount of substance decomposed in time t.

Then, by Guldberg and Waage's law

$$\frac{dy}{dt} = \mathrm{K}(\mathrm{A}-x)(\mathrm{B}-y).$$

Now it has been shown experimentally that the decomposition of a ferment progresses as a monomolecular reaction. Hence, if c = velocity constant of the decomposition of the ferment,

$$c = \frac{1}{t}\log_e\frac{\mathrm{A}}{\mathrm{A}-x},$$

i.e.

$$e^{ct} = \frac{\mathrm{A}}{\mathrm{A}-x},$$

or

$$\mathrm{A}-x = \frac{\mathrm{A}}{e^{ct}} = \mathrm{A}e^{-ct}.$$

Hence our equation becomes

$$\frac{dy}{dt} = \mathrm{KA}e^{-ct}(\mathrm{B}-y),$$

or

$$\int\frac{dy}{\mathrm{B}-y} = \mathrm{KA}\int e^{-ct}dt,$$

i.e.

$$-\log_e(\mathrm{B}-y) = -\frac{\mathrm{KA}}{c}e^{-ct}+\mathrm{C}.$$

When

$$t = 0, \quad y = 0,$$

$$\therefore \quad -\log_e \mathrm{B} = -\frac{\mathrm{KA}}{c}+\mathrm{C},$$

whence

$$\mathrm{C} = \frac{\mathrm{KA}}{c} - \log_e \mathrm{B}.$$

$$\therefore \quad \log_e(\mathrm{B}-y) = \frac{\mathrm{KA}}{c}e^{-ct} - \frac{\mathrm{KA}}{c} + \log_e \mathrm{B}.$$

$$\therefore \quad \log_e\frac{(\mathrm{B}-y)}{\mathrm{B}} = \frac{\mathrm{KA}}{c}(e^{-ct}-1).$$

Hence after an indefinite period, *i.e.* when $t = \infty$, we have

$$\log_e\frac{(\mathrm{B}-y)}{\mathrm{B}} = -\frac{\mathrm{KA}}{c}$$

$$\therefore \quad \frac{\mathrm{B}-y}{\mathrm{B}} \text{ or } 1-\frac{y}{\mathrm{B}} = e^{-\frac{\mathrm{KA}}{c}}$$

$$\therefore \quad y = \mathrm{B}\left(1-e^{-\frac{\mathrm{KA}}{c}}\right),$$

i.e. the amount of substance (y) transformed after any period of time is always less by $Be^{-\frac{KA}{c}}$ than the amount originally present.

(5) The empirical Schütz-Borissoff law, $x = K\sqrt{Fat}$ (p. 179), leads on differentiation to the equation $\frac{dx}{dt} = \frac{kFa}{x}$ $\left(\text{where } k = \frac{K^2}{2}\right)$. As, according to Arrhenius, $\frac{F}{x}$ is proportional to the concentration of the enzyme at any moment, and as $(a - x)$ is the concentration of the protein at the same moment, we have (by the law of mass action)

$$\frac{dx}{dt} = \frac{KF}{x}(a - x),$$

$$\therefore \quad KFt = \int\frac{x}{a - x}dx = a \log_e \frac{1}{a - x} - x + C \text{ (see p. 307).}$$

But when $t = 0$, $x = 0$, therefore $a \log_e \frac{1}{a} + C = 0$, whence $C = a \log_e a$,

$$\therefore \quad KFt = a \log_e \frac{1}{a - x} - x + a \log_e a = a \log_e \frac{a}{a - x} - x.$$

$$\therefore \quad \frac{1}{t}\left[a \log_e \frac{a}{a - x} - x\right] = KF = \text{constant.}$$

This theoretical formula for peptic digestion is totally different from the empirical formula $x = K\sqrt{Fat}$, or $x/\sqrt{t}$ = constant. Nevertheless each formula gives satisfactory results up to a certain stage of digestion. This is so because in the early stages, when x (the amount of protein digested) is small compared with a (the original amount of protein), $(a - x)$ does not differ materially from a, so that $\frac{dx}{dt} = \frac{KF}{x}(a - x)$ is practically the same as $\frac{dx}{dt} = \frac{kFa}{2x}$, which leads on integration to the empirical Schütz-Borissoff law.

The foregoing example is very instructive for the following reasons:—

(1) *It is an illustration of the fact that two totally different formulæ may*—within a certain range of values of the independent variable—*equally express the relationship between the two variables.* Thus, for values of x up to $x = 33{\cdot}7$ per cent. either of the two formulæ will give equally satisfactory results, as the table over page shows.

But while the complete logarithmic formula gives practically uniform results for all values of x, with the Schütz-Borissoff formula the results are not uniform after $x = 33{\cdot}7$ per cent.

(2) *It shows how risky it is to use an empirical formula for purposes of extrapolation, i.e.* for calculating values of the dependent variable outside the range of observation for values

Time (t) (in hours).	Protein Digested. (x.)	$\frac{1}{t}\left\{a \log_e \frac{a}{a-x} - x\right\}$ = constant.	$\frac{x}{\sqrt{t}}$ = constant.
	Per cent.		
2	10·5	3·0	7·5
4	16·4	3·8	8·2
6	19·9	3·8	8·1
8	22·7	3·8	8·0
12	27·0	3·7	7·7
16	30·4	3·6	7·6
20	33·7	3·7	7·5
32	40·0	3·4	7·1
48	45·1	3·2	6·5
64	50·8	3·1	6·3
96	57·4	2·8	5·9

of the independent variable. Thus, although the Schütz-Borissoff formula may be safely employed for purposes of interpolation, *i.e.* for calculating values of the dependent variable for values *within* the range of observation of x, yet if used for extrapolation it would give discordant results (see, further, Chapter XXII., p. 357).

(6) The rate of multiplication of micro-organisms at any moment in the presence of a *limited supply* of nutriment, such as obtains in test-tube experiments, is proportional both to the number of organisms as well as to the concentration of the foodstuff at that moment. Assuming that the organisms multiply by a simple conversion of available food material into other organisms, find an equation for computing the number of organisms y present after an interval of time t (hours), taking the original number of organisms (*i.e.* those present at time $t = 0$) to be y_0. (A. G. M'Kendrick and M. Kesava Pai, *Proc. R. Soc. Edin.*, xxxi., 1911.)

Let a = original concentration of foodstuff.
(Since y_0 is very small compared with y_∞ or a, $a - y_0$ may be written a.)
Then, by hypothesis,

$$a - y = \text{concentration of foodstuff at any moment } t,$$

and $\frac{dy}{dt} = by(a-y)$, where b is a constant representing the ability of the organism to assimilate its food. (i)

$$\therefore \quad \frac{dy}{y(a-y)} = bdt.$$

Splitting into partial fractions we get

$$\frac{dy}{y(a-y)} = \frac{1}{a}\left\{\frac{dy}{y} + \frac{dy}{a-y}\right\},$$

$$\therefore \quad \frac{1}{a}\int\frac{dy}{y} + \frac{1}{a}\int\frac{dy}{a-y} = b\int dt,$$

or $$\frac{1}{a}\log_e y - \frac{1}{a}\log_e (a-y) = bt + C. \quad (C = \text{integration constant.})$$

$\therefore\ \log_e \dfrac{y}{a-y} = abt + aC$ (compare similar equation in connection with human growth, p. 350).

To evaluate C put $t = 0$, when y becomes y_0 and $a-y$ becomes $a-y_0$.

$$\therefore\quad \log_e \frac{y_0}{a-y_0} = aC,$$

$$\therefore\qquad C = \frac{1}{a}\log_e \frac{y_0}{a-y_0}.$$

Hence the complete equation becomes

$$\log_e \frac{y}{a-y} = abt + \log_e \frac{y_0}{a-y_0}$$

or $$\log_e \frac{y(a-y_0)}{y_0(a-y)} = abt$$

$$\therefore\qquad \frac{y(a-y_0)}{y_0(a-y)} = e^{abt}$$

i.e. $$ya - yy_0 = y_0ae^{abt} - yy_0e^{abt}$$

$$\therefore\quad y[a + y_0e^{abt} - y_0] = y_0ae^{abt}$$

$$\therefore\qquad y = \frac{y_0ae^{abt}}{a - y_0 + y_0e^{abt}}$$

$$= \frac{a}{\dfrac{a-y_0}{y_0}e^{-abt} + 1}. \qquad . \qquad . \qquad \text{(ii)}$$

Notes.—(1) When the organism stops increasing $\dfrac{dy_\infty}{dt} = 0$ (where y_∞ stands for the number of organisms when growth ceases),

$$\therefore\quad \text{from equation (i), } (a - y_\infty) = 0,$$

$$i.e.\quad y_\infty = a.$$

Hence a = number of organisms when growth ceases.

(2) Equation (ii) can now be simplified in the case of rapidly growing bacteria where a is thousands of times as large as y_0. For then $a - y_0$ is practically the same as a, and therefore the equation becomes

$$y_t = \frac{a}{\dfrac{a}{y_0}e^{-abt} + 1}$$

(3) For evaluation of b, see Example (6), p. 363.

(7) **The Hæmodynamics of Mitral and Tricuspid Incompetence.**—A heart with a leaking auriculo-ventricular valve is to some extent analogous to an open vessel (which, owing to the comparatively small size of the ventricular cavity, may—without appreciable error—be taken as cylindrical, instead of conical) receiving fluid (blood) at a uniform rate at the top

(through the open auriculo-ventricular valve during diastole), and losing fluid (blood) at a uniform rate at the bottom (through the leaking valve during the ejection or sphygmic period of systole). In a normal non-leaking heart the isometric or presphygmic period of systole, *i.e.* the time taken by the ventricle to acquire sufficient pressure to overcome the diastolic pressure in the aorta (75 mm. mercury), or pulmonary artery (about 20 mm. mercury), and force open the semilunar valves, is 0·05 second (the total period of systole occupying 0·3 second). How long will this presphygmic period be in a heart with a regurgitant mitral or tricuspid valve, respectively, with a leak of a square centimetres, if the sectional area of the auriculo-ventricular orifice in diastole is 8 square centimetres? How big must the leak be for the ventricle to fail to force the blood through the semilunar valves? (*Cf.* Emil Schwarz, "*Wiener Klin. Wochenschr.*," 1905.)

Suppose the bottom of the cylindrical vessel to have its orifice plugged (representing normal heart); then if the pressure of the fluid increases at the rate of p metres water per second, it will increase by the amount pdt during the interval dt, whilst the level of the fluid during the same interval will rise by an amount dx. Therefore $dx = pdt$. Now suppose the bottom orifice, whose sectional area is a square centimetres, to be opened (leaking valve); then an amount of fluid equal to, say, dQ will flow out of the leak during the time dt, and the fall of level due to this loss will be $\frac{dQ}{A}$ (where A = sectional area of the cylinder, = 8 square centimetres in the case of the heart).

$$\therefore \quad dx = pdt - \frac{dQ}{A}.$$

But, by Torricelli's theorem, $\frac{dQ}{A} = \mu a\sqrt{\frac{2gx}{A^2 - a^2}}dt$

(where μ = coefficient of discharge = 0·62, and g = acceleration due to gravity = 9·81 metres per second per second)

$$= 0{\cdot}62a\sqrt{\frac{19{\cdot}6}{64 - a^2}}\sqrt{x}\,.\,dt$$

$$= \frac{2{\cdot}7464a\sqrt{x}}{\sqrt{64 - a^2}}dt.$$

$$\therefore \quad dx = \left(p - \frac{2{\cdot}7464a\sqrt{x}}{\sqrt{64 - a^2}}\right)dt$$

$$= \left(p - \frac{1}{R}\sqrt{x}\right)dt, \text{ where } R = \frac{\sqrt{64 - a^2}}{2{\cdot}7464a}$$

$$= \frac{1}{R}(pR - \sqrt{x})dt.$$

Separating the variables and integrating, we obtain

$$t\left(\text{or} \int dt\right) = R\int\frac{dx}{pR - \sqrt{x}} = 2R\left[pR\log_e\frac{1}{pR - \sqrt{x}} - \sqrt{x}\right] + C \text{ (see p. 307).}$$

As the diastolic intraventricular pressure is zero and the systolic intraventricular pressure at the end of the presphygmic period is 75 mm. mercury

(1 metre water) in the left ventricle, and about 20 mm. mercury (about 0·25 metre water) in the right ventricle, we therefore have:

Presphygmic period, t, for left ventricle

$$= 2R\left[pR \log_e \frac{1}{pR - \sqrt{x}} - \sqrt{x}\right]_0^1$$

$$= 2R\left[2{\cdot}302585pR \log_{10} \frac{pR}{pR - 1} - 1\right]$$

and presphygmic period, t, for right ventricle

$$= 2R\left[2{\cdot}302585pR \log_{10} \frac{1}{pR - \sqrt{x}} - \sqrt{x}\right]_0^{0\cdot25}$$

$$= 2R\left[2{\cdot}302585pR \log_{10} \frac{pR}{pR - 0{\cdot}5} - 0{\cdot}5\right]$$

As in the normal heart the change of intraventricular pressure from 0 to 1 metre water, or from 0 to 0·25 metre water (in the left and right ventricles respectively), occurs in 0·05 second (= normal presphygmic period), therefore p, which is the change of intraventricular pressure per second, = 20 and 5 metres water respectively. Hence, as $R = \frac{\sqrt{64 - a^2}}{2{\cdot}7464a}$, we have all the data for computing the presphygmic period, t, in the case of a leaking auriculo-ventricular valve, for any given size, a, of the leak. Conversely, if on a sufficiently magnified tracing of the apex beat—as obtained by means of a polygraph—one were able to measure accurately the duration of the presphygmic period (the space between Mackenzie's points 2 and 3), one would be able to estimate the size of the leak in any given case of auriculo-ventricular incompetence on the supposition, of course, that the normal presphygmic period is accurately known.

Using seven-figure logarithms, we obtain the following table (*cf.* Example (2), p. 11):—

a.	$R = \frac{\sqrt{64 - a^2}}{2{\cdot}7464a}$.	Mitral Regurgitation (p = 20 metres water).		Tricuspid Regurgitation (p = 5 metres water)	
		t.	Increase of t (per cent.).	t.	Increase of t (per cent.).
1	2·890134	0·0505	1·0	0·0512	2·4
2	1·410241	0·0512	2·4	0·0525	5·0
3	0·900135	0·0520	4·0	0·0540	8·0
4	0·630679	0·0526	5·2	0·0560	12·0
..	..	..	..	..	..
7	0·201463	0·0602	20·4	0·0769	53·8
7·5	0·135157	0·0672	34·4	0·0817	63·4
7·6	0·110190	0·0733	46·6	0·1789	257·8
7·8	0·082980	0·0882	76·4	∞	∞
7·9	0·058119	∞	∞		

The presphygmic period, therefore, increases with the size of the leak—slowly at first, but more and more rapidly with increase of a. When a reaches such a size as to make $\log_e \frac{1}{pR - \sqrt{x}}$ equal to ∞, t becomes ∞, so that the ventricle is, in that case, never able to work up enough pressure to open the semilunar valves. This occurs when $pR - \sqrt{x} = 0$, *i.e.* when R or $\frac{\sqrt{64 - a^2}}{2 \cdot 7464a} = \frac{\sqrt{x}}{p} = 0 \cdot 05$ (in mitral regurgitation) or $\frac{0 \cdot 5}{5} = 0 \cdot 1$ (in tricuspid regurgitation).

Hence $t = \infty$ in mitral regurgitation when $a = 7 \cdot 9$ square centimetres and in tricuspid regurgitation when $a = 7 \cdot 7$ square centimetres.

Note.—Although theoretically it is always possible for a leaking heart in which the size of the leak is a fraction of a square centimetre less than that of the normal open auriculo-ventricular valvular orifice to pump blood through the semilunar valves, nevertheless some compensatory process must occur to overcome the effects of the prolonged presphygmic period—if the circulation is to continue. For the slightest *increase* in the presphygmic period necessarily entails an equal *decrease* in the sphygmic or ejection period (if the total systolic period is to remain unaltered at 0·3 second), with a corresponding diminution in the stroke-volume (*i.e.* the amount of blood pumped out during each systole), leaving a progressively increasing residuum within the ventricle. This is shown in the following table, in which V stands for the volume of blood received by the ventricle from the auricle during each diastole and a represents a fraction.

Number of Systole (from beginning of leak).	Content of Ventricle.	Stroke-volume.	Residuum in Ventricle.
1st . . .	V	$V(1-a)$	aV
2nd . . .	$V(1+a)$	$V(1-a)$	$2aV$
3rd . . .	$V(1+2a)$	$V(1-a)$	$3aV$
. . .	. . .	. . .	. . .
nth . . .	$V[1+(n-1)a]$	$V(1-a)$	naV

Let a be very small, say $\frac{1}{10{,}000}$. If the ventricle is to receive the full auricular output V at each diastole, it will have to increase rapidly in size, and during 10,000 systoles (*i.e.* after about 2 hours), when the residual blood within it has become equal to V, the ventricle will have to become double in size. During another 2 hours its size will have to treble. Thus the ventricle will have to continue to increase indefinitely in arithmetical progression every 2 hours. This impossible state of affairs is overcome by hypertrophy of the ventricular musculature, which enables the ventricle to expel its normal blood content in a shorter period.

For further examples in differential equations with separable variables the reader is referred to the various problems in connection with chemical kinetics as well as the physiology of growth that have been considered on previous pages.

CHAPTER XXI.

FOURIER'S THEOREM.

We have seen that Maclaurin's theorem enables us to expand certain functions of x into series of ascending powers of x. If we have to deal with a *periodic* function, then Fourier's theorem enables us to expand any such function into a series of sines and cosines of multiples of the independent variable.

Thus, while Maclaurin's theorem states that

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots,$$

Fourier's theorem in the case of periodic functions states that

$$f(x) = A_0 + (A_1 \sin x + B_1 \cos x) + (A_2 \sin 2x + B_2 \cos 2x) \\ + (A_3 \sin 3x + B_3 \cos 3x) + \ldots$$

This theorem, apart from its enormous importance in all branches of physics, is of interest to the student of the physiology of hearing. It expresses the fact that every musical sound, such as a violin note, may always be resolved into a number of simple tones corresponding to the fundamental and its partials, and Helmholtz's theory of hearing assumes that the various radial fibres of the basilar membrane and the corresponding arches of Corti are excited only by those partials of the compound sound to which the fibres are tuned.

To Evaluate the Constants A_0, A_1, . . . A_n, B_1, B_2, . . . B_n.—Since, during a complete period, x changes from 0 to 2π, therefore we integrate both sides of the equation between the limits 0 and 2π.

Thus

$$\int_0^{2\pi} f(x)dx = A_0\int_0^{2\pi} dx + A_1\int_0^{2\pi}\sin x dx + A_2\int_0^{2\pi}\sin 2x dx + \ldots$$

$$+ B_1\int_0^{2\pi}\cos x dx + B_2\int_0^{2\pi}\cos 2x dx + \ldots$$

Now $$\int_0^{2\pi}\sin nx dx = \left[-\frac{1}{n}\cos nx\right]_0^{2\pi} = -\frac{1}{n} + \frac{1}{n} = 0.$$

$$\therefore \qquad \int_0^{2\pi} \sin x dx = 0,$$

$$\int_0^{2\pi} \sin 2x dx = 0,$$

etc.

Similarly $\int_0^{2\pi} \cos nx dx = \left[\frac{1}{n} \sin nx\right]_0^{2\pi} = 0.$

$$\therefore \qquad \int_0^{2\pi} \cos x dx = 0,$$

$$\int_0^{2\pi} \cos 2x dx = 0,$$

etc.

$$\therefore \qquad \int_0^{2\pi} f(x) dx = A_0 \int_0^{2\pi} dx = 2\pi A_0.$$

$$\therefore \qquad A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} y dx.$$

$\therefore$ When $\int_0^{2\pi} y dx$ is known, A_0 is known.

To find the values of A_n and B_n, multiply each side of the equation by $\sin nx$ and $\cos nx$ respectively and integrate.

Thus
$$\int_0^{2\pi} f(x) \sin nx dx = A_0 \int_0^{2\pi} \sin nx dx + A_1 \int_0^{2\pi} \sin nx \sin x dx$$
$$+ \ldots + A_n \int_0^{2\pi} \sin^2 nx dx + \ldots$$
$$+ B_1 \int_0^{2\pi} \sin nx \cos x dx + B_2 \int_0^{2\pi} \sin nx \cos 2x dx$$
$$+ \ldots + B_n \int_0^{2\pi} \sin nx \cos nx dx + \ldots$$

But
$$\int_0^{2\pi} \sin nx dx = 0 \text{ (see above)},$$

and from p. 50 we know that

$$\int_0^{2\pi} \sin nx \sin x dx = \tfrac{1}{2} \int_0^{2\pi} \cos (n-1) x dx - \tfrac{1}{2} \int_0^{2\pi} \cos (n+1) x dx = 0.$$

Similarly for all the other terms except $\int_0^{2\pi} \sin^2 nx dx$, which is equal to

$$\tfrac{1}{2}\int_0^{2\pi}(1-\cos 2nx)dx = \tfrac{1}{2}\int_0^{2\pi}dx - \tfrac{1}{2}\int_0^{2\pi}\cos 2nx dx = \pi - 0 = \pi.$$

$$\therefore \quad \int_0^{2\pi} f(x)\sin nx dx = A_n\pi.$$

$$\therefore \qquad A_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} y\sin nx dx.$$

Similarly,
$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} y\cos nx dx.$$

Hence
$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y\sin x dx; \quad A_2 = \frac{1}{\pi}\int_0^{2\pi} y\sin 2x dx; \quad \text{etc.}$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y\cos x dx; \quad B_2 = \frac{1}{\pi}\int_0^{2\pi} y\cos 2x dx; \quad \text{etc.}$$

Note: That $\int_0^{2\pi}\sin nx dx$ and $\int_0^{2\pi}\cos nx dx$ are each equal to zero is easily seen by looking at the sine and cosine curves (p. 127). For each period the area of the portion below the abscissæ is equal and negative to that above.

Graphical Method of Fourier Analysis.—We shall now make use of these formulæ for the purpose of **Harmonic Analysis,** *i.e.* **the analysis of a periodic curve into its component curves.**

Suppose we are given a curve (fig. 126) representing some harmonic vibration—say, a sound wave—and we wish to find its equation or its component waves, viz. the fundamental and the partials. To simplify matters, suppose we are told that the wave consists only of the fundamental and its first harmonic.

If we call the fundamental x and the 1st harmonic $2x$ then the equation of the curve will be

$$y = f(x) = A_0 + A_1\sin x + A_2\sin 2x + B_1\cos x + B_2\cos 2x.$$

The problem is to evaluate the constants A_0, A_1, A_2, and B_1, B_2. This can be done graphically by the mid-ordinate method (p. 296).

The base of the whole curve is the period $2\pi = 360°$. Divide the base into, say, ten equal parts representing the angles 36°, 72°, 108°, etc., as shown in the figure, and erect the mid-ordinates $y_1, y_2, y_3, \ldots y_{10}$ at the points corresponding to the angles 18°, 54°, 90°, etc. Measure these mid-ordinates (*i.e.* the

ordinates at $x = 18°, 54°, 90°$, etc., respectively) and tabulate their values as follows:—

$y_1 = 1·56$	$y_6 = -1·13$
$y_2 = 3·75$	$y_7 = -2·91$
$y_3 = 4$	$y_8 = -4$
$y_4 = 2·91$	$y_9 = -3·75$
$y_5 = 1·13$	$y_{10} = -1·56$

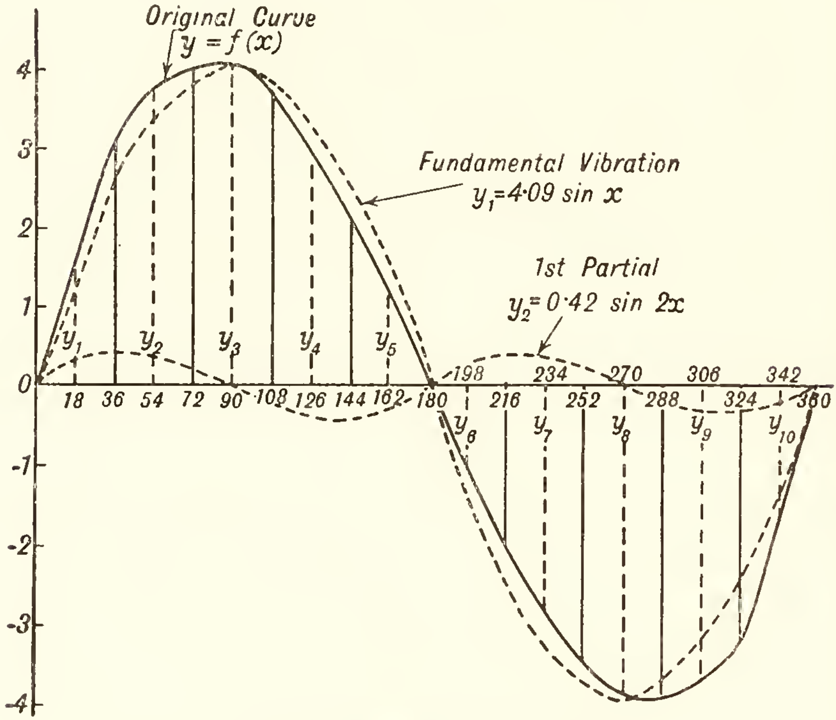


Fig. 126.—Harmonic Analysis.

Then since the sum of these ordinates = 0,

$\therefore \quad \int_0^{2\pi} ydx$ [which, of course, corresponds to $(y_1 + y_2 + \ldots + y_{10})$] $= 0$.

$\therefore \quad A_0$ which equals $\frac{1}{2\pi}\int_0^{2\pi} ydx = 0$.

$\therefore \quad$ The equation of the curve is

$$y = A_1 \sin x + A_2 \sin 2x + B_1 \cos x + B_2 \cos 2x.$$

Now, $A_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin xdx$,

which is approximately the same as

$$\frac{1}{\pi}\left[y_1 \sin 18° + y_2 \sin 54° + y_3 \sin 90° + \ . \ . \ . \ + y_{10} \sin 342°\right]$$

$$= \frac{1}{\pi}\Big[1{\cdot}56 \times 0{\cdot}309 + 3{\cdot}75 \times 0{\cdot}809 + 4 \times 1 + 2{\cdot}91 \times 0{\cdot}809 + 1{\cdot}13 \times$$
$$0{\cdot}309 + (-1{\cdot}13)(-0{\cdot}309) + (-2{\cdot}91)(-0{\cdot}809) + (-4)(-1) +$$
$$(-3{\cdot}75)(-0{\cdot}809) + (-1{\cdot}56)(-0{\cdot}309).\Big] \quad \text{(See table below.)}$$

$$= \frac{1}{\pi}[0{\cdot}309(1{\cdot}56 + 1{\cdot}13 + 1{\cdot}13 + 1{\cdot}56) + 4(1+1) + 0{\cdot}809(3{\cdot}75 + 2{\cdot}91 + 2{\cdot}91 + 3{\cdot}75)]$$

$$= \frac{1}{\pi}[0{\cdot}309(3{\cdot}12 + 2{\cdot}26) + 8 + 0{\cdot}809(7{\cdot}5 + 5{\cdot}82)]$$

$$= \frac{1}{\pi}(0{\cdot}309 \times 5{\cdot}38 + 8 + 0{\cdot}809 \times 13{\cdot}32]$$

$$= \frac{1}{\pi}(1{\cdot}662 + 8 + 10{\cdot}776)$$

$$= \frac{1}{\pi} \times 20{\cdot}44.$$

But since 10 parts of the base $= 2\pi$,

$$\therefore \quad \pi = 5 \text{ parts.}$$
$$\therefore \quad A_1 = \tfrac{1}{5} \times 20{\cdot}44 = 4{\cdot}09.$$

To facilitate the work the values are tabulated as follows:—

y	x	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$
$y_1 = 1{\cdot}56$	18°	0·309	0·951	0·588	0·809
$y_2 = 3{\cdot}75$	54°	0·809	0·588	0·951	−0·309
$y_3 = 4$	90°	1	0	0	−1
$y_4 = 2{\cdot}91$	126°	0·809	−0·588	−0·951	−0·309
$y_5 = 1{\cdot}13$	162°	0·309	−0·951	0·588	0·809
$y_6 = -1{\cdot}13$	198°	−0·309	−0·951	0·588	0·809
$y_7 = -2{\cdot}91$	234°	−0·809	−0·588	0·951	−0·309
$y_8 = -4$	270°	−1	0	0	−1
$y_9 = -3{\cdot}75$	306°	−0·809	0·588	−0·951	−0·309
$y_{10} = -1{\cdot}56$	342°	−0·309	0·951	−0·588	0·809

Similarly, $A_2 = \frac{1}{\pi}\int_0^{2\pi} y \sin 2x dx.$

$$= \tfrac{1}{5}[y_1 \sin 36^\circ + y_2 \sin 108^\circ + y_3 \sin 180^\circ + \ldots + y_{10} \sin 684^\circ]$$

$$= \tfrac{1}{5}[1{\cdot}56 \times 0{\cdot}588 + 3{\cdot}75 \times 0{\cdot}951 + 4 \times 0 + 2{\cdot}91(-0{\cdot}951) + 1{\cdot}13(-0{\cdot}588) + (-1{\cdot}13) \times 0{\cdot}588 + (-2{\cdot}91) \times 0{\cdot}951 + (-4) \times 0 + (-3{\cdot}75)(-0{\cdot}951) + (-1{\cdot}56)(-0{\cdot}588)]$$

$$= \tfrac{1}{5}[0{\cdot}588(1{\cdot}56 - 1{\cdot}13 - 1{\cdot}13 + 1{\cdot}56) + 0{\cdot}951(3{\cdot}75 - 2{\cdot}91 - 2{\cdot}91 + 3{\cdot}75)]$$

$$= \tfrac{1}{5}(0{\cdot}588 \times 0{\cdot}86 + 0{\cdot}951 \times 1{\cdot}68).$$

$$= \tfrac{1}{5} \times 2{\cdot}103$$

$$= 0{\cdot}421.$$

Similarly, $B_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos x dx$

$$= \tfrac{1}{5}[0{\cdot}951(1{\cdot}56 - 1{\cdot}13 + 1{\cdot}13 - 1{\cdot}56) + 0{\cdot}588(3{\cdot}75 - 2{\cdot}91 + 2{\cdot}91 - 3{\cdot}75)]$$

$$= 0,$$

and $B_2 = \tfrac{1}{5}[0{\cdot}809(1{\cdot}56 + 1{\cdot}13 - 1{\cdot}13 - 1{\cdot}56) + 1(-4+4) + 0{\cdot}309(-3{\cdot}75 - 2{\cdot}91 + 2{\cdot}91 + 3{\cdot}75)]$

$$= 0.$$

$\therefore$ The equation contains no *cos* terms.

Hence we get the final equation

$$y = 4{\cdot}09 \sin x + 0{\cdot}42 \sin 2x \text{ (see fig. 126).}$$

In other words, the components of the function

are $y_1 = 4{\cdot}09 \sin x,$

and $y_2 = 0{\cdot}42 \sin 2x.$

EXAMPLE.

Analyse the curve resulting from the following plotting table:—

x	0	45°	90°	135°	180°	225°	270°	315°	360°
y	0	21·5	31·25	11·25	0	9	30	26·5	0

assuming it to consist of the fundamental and the first two harmonics.

The curve is shown in fig. 127. If we draw the mid-ordinates and measure them we can tabulate the values as follows:—

y	x	sin x	cos x	sin $2x$	cos $2x$	sin $3x$	cos $3x$
$y_1 = 12$	22·5	0·3827	0·9239	0·7071	0·7071	0·9239	+0·3827
$y_2 = 27·75$	67·5	0·9239	0·3827	0·7071	−0·7071	−0·3827	−0·9239
$y_3 = 23$	112·5	0·9239	−0·3827	−0·7071	−0·7071	−0·3827	0·9239
$y_4 = 5·25$	157·5	0·3827	−0·9239	−0·7071	0·7071	0·9239	−0·3827
$y_5 = 3$	202·5	−0·3827	−0·9239	0·7071	0·7071	−0·9239	−0·3827
$y_6 = 23·5$	247·5	−0·9239	−0·3827	0·7071	−0·7071	0·3827	0·9239
$y_7 = 28·75$	292·5	−0·9239	−0·3827	−0·7071	−0·7071	0·3827	−0·9239
$y_8 = 19·5$	337·5	−0·3827	0·9239	−0·7071	0·7071	−0·9239	0·3827

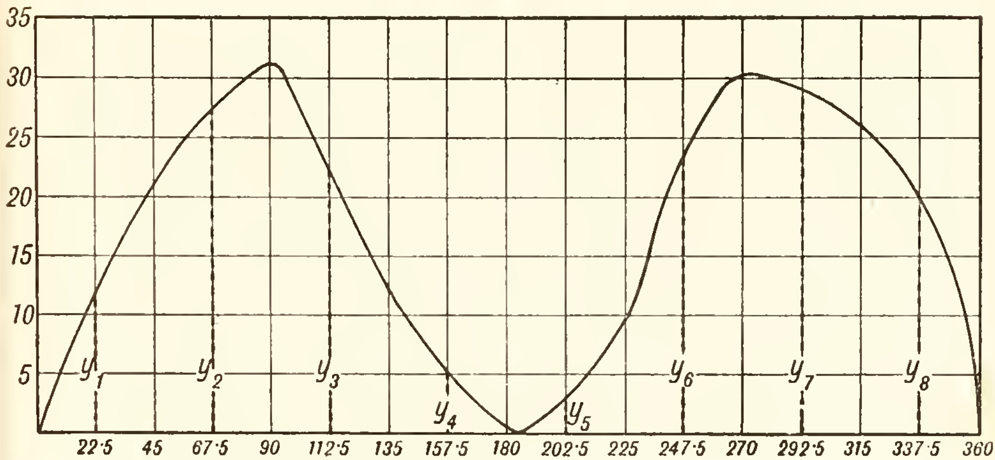


FIG. 127.—Another Example of Harmonic Analysis.

$\therefore$ Value of sum of mean ordinates $= \Sigma y = 142·75$,

i.e. $$A_0 = \frac{142·75}{8} = 17·84.$$

$$\begin{aligned} A_1 = \tfrac{1}{4}\Sigma y \sin x &= \tfrac{1}{4}[0·3827(12 + 5·25 - 3 - 19·5) \\ &\qquad + 0·9239(27·75 + 23 - 23·5 - 28·75)] \\ &= \tfrac{1}{4}[0·383(-5·25) + 0·924(-1·50)] \\ &= -0·85. \end{aligned}$$

$$\begin{aligned} A_2 = \tfrac{1}{4}\Sigma y \sin 2x &= \tfrac{1}{4}[0·71(12 + 27·75 + 3 \\ &\qquad + 23·5 - 23 - 5·25 - 28·75 - 19·5)] \\ &= 0·178(66·25 - 76·5) = -1·8. \end{aligned}$$

$$\begin{aligned} A_3 = \tfrac{1}{4}\Sigma y \sin 3x &= \tfrac{1}{4}[0·924(12 + 5·25 - 3 - 19·5) \\ &\qquad + 0·383(23·5 + 28·75 - 27·75 - 23)] \\ &= \tfrac{1}{4}[0·924(-5·25) + 0·383(1·5)] \\ &= -1·07. \end{aligned}$$

$$
\begin{aligned}
B_1 = \tfrac{1}{4}\Sigma y \cos x &= \tfrac{1}{4}[0{\cdot}924(12-5{\cdot}25-3+19{\cdot}5) \\
&\qquad +0{\cdot}383(27{\cdot}75-23-23{\cdot}5+28{\cdot}75)] \\
&= \tfrac{1}{4}(0{\cdot}924\times 23{\cdot}25+0{\cdot}383\times 10) \\
&= 6{\cdot}3. \\
B_2 = \tfrac{1}{4}\Sigma y \cos 2x &= \tfrac{1}{4}\times 0{\cdot}707(12-27{\cdot}75-23+5{\cdot}25 \\
&\qquad +3-23{\cdot}5-28{\cdot}75+19{\cdot}5) \\
&= -11{\cdot}2. \\
B_3 = \tfrac{1}{4}\Sigma y \cos 3x &= \tfrac{1}{4}[0{\cdot}383(12-5{\cdot}25-3+19{\cdot}5) \\
&\qquad +0{\cdot}924(23+23{\cdot}5-27{\cdot}75-28{\cdot}75)] \\
&= \tfrac{1}{4}[0{\cdot}383\times 23{\cdot}25-0{\cdot}924\times 10] \\
&= -0{\cdot}085.
\end{aligned}
$$

Hence, the equation of the curve is

$$
\begin{aligned}
y = 17{\cdot}84 &- 0{\cdot}85 \sin x - 1{\cdot}8 \sin 2x - 1{\cdot}07 \sin 3x \\
&+ 6{\cdot}3 \cos x - 11{\cdot}2 \cos 2x - 0{\cdot}085 \cos 3x.
\end{aligned}
$$

Note.—The number of ordinates given is not sufficient to ensure accurate results.

Composition of Harmonic Motions.—The reverse process of finding the resultant curve when its components are known is very easy. Thus, supposing we were given that the two components are

$$y_1 = 4{\cdot}09 \sin x, \quad y_2 = 0{\cdot}42 \sin 2x,$$

we tabulate as follows:—

x	0	18°	54°	90°	126°	162°	198°	234°	270°	306°	342°
sin x	0	0·309	0·809	1	0·809	0·309	−0·309	−0·809	−1	−0·809	−0·309
sin $2x$	0	0·588	0·951	0	−0·951	−0·588	0·588	0·951	0	−0·951	−0·588
y_1	0	1·264	3·309	4·09	3·309	1·264	−1·264	−3·309	−4·09	−3·309	−1·264
y_2	0	0·250	0·404	0	−0·404	−0·250	0·250	0·404	0	−0·404	−0·250
$y = y_1+y_2$	0	1·514	3·713	4·09	2·905	1·014	−1·014	−2·905	−4·09	−3·713	−1·514

If we plot the three curves from the values of y_1, y_2 and y, we get on the same diagram the two components and their resultant curve (see fig. 126).

If the actual graphs of the two components are given, then the resultant curve is quickly drawn by adding **(algebraically)** the ordinates at various points on the abscissa. Thus, at the abscissal point 54°, the ordinate of the resultant curve is **arithmetical sum** of y_1 and y_2, and at abscissal point 126°, the ordinate of the resultant is equal to the **arithmetical difference** of y_1 and y_2.

Harmonic Analysers.—There are a number of instruments which determine mechanically the coefficients in a Fourier Series, in the same way as a planimeter determines mechanically the area enclosed by any given curve. Such instruments of course save the enormous labour of calculation, but a description of them is outside the scope of this book.

EXAMPLE.

It has been shown that the temperature of a normal person varies with the time of day, and that when the 24 hourly observations are plotted, the curve obtained forms a cycle, which repeats itself regularly every 24 hours (see fig. 25, p. 98). Analyse the curve by the Fourier method.

Divide the base of the whole cycle of 24 hours (*i.e.* 2π) into 12 equal portions at the hours 2, 4, 6, etc. Each portion will then represent 30°.

Now take the mid-ordinates, *i.e.* the temperatures y_1, y_2, y_3, etc., at the hours 1, 3, 5, etc. (*i.e.* at 15°, 45°, 75°, 105°, etc.), when the values of the constants $\frac{1}{2\pi}\int_0^{2\pi} y\,dx$, $\frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx$, $\frac{1}{\pi}\int_0^{2\pi} y \cos nx\,dx$, may with sufficient accuracy be taken as $\frac{1}{12}\Sigma y$, $\frac{1}{6}\Sigma y \sin nx$, $\frac{1}{6}\Sigma y \cos nx$, respectively.

Let us stop at $3x$ and represent the curve by the equation

$$y = A_0 + A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x.$$

Then $A_0 = \frac{1}{12}\Sigma y = \frac{1}{12}[36{\cdot}55 + 36{\cdot}46 + \ldots + 36{\cdot}68] = 36{\cdot}845.$

$$\begin{aligned} A_1 = \tfrac{1}{6}\Sigma y \sin x &= \tfrac{1}{6}[0{\cdot}259(36{\cdot}55 + 36{\cdot}83 - 37{\cdot}09 - 36{\cdot}68) \\ &\quad + 0{\cdot}707(36{\cdot}46 + 36{\cdot}72 - 37{\cdot}30 - 36{\cdot}92) \\ &\quad + 0{\cdot}966(36{\cdot}48 + 36{\cdot}62 - 37{\cdot}32 - 37{\cdot}17)] \\ &= -\tfrac{1}{6}(0{\cdot}101 + 0{\cdot}735 + 1{\cdot}343) = -0{\cdot}363. \end{aligned}$$

$$\begin{aligned} A_2 = \tfrac{1}{6}\Sigma y \sin 2x &= \tfrac{1}{6}[0{\cdot}5(36{\cdot}55 + 36{\cdot}48 - 36{\cdot}62 - 36{\cdot}83 \\ &\quad + 37{\cdot}09 + 37{\cdot}32 - 37{\cdot}17 - 36{\cdot}68) \\ &\quad + 1(36{\cdot}46 - 36{\cdot}72 + 37{\cdot}30 - 36{\cdot}92)] \\ &= \tfrac{1}{6}(0{\cdot}07 + 0{\cdot}12) = +0{\cdot}032. \end{aligned}$$

$$\begin{aligned} A_3 &= \tfrac{1}{6}[0{\cdot}707(36{\cdot}55 + 36{\cdot}46 - 36{\cdot}48 - 36{\cdot}62 + 36{\cdot}72 + 36{\cdot}83 \\ &\quad - 37{\cdot}09 - 37{\cdot}30 + 37{\cdot}32 + 37{\cdot}17 - 36{\cdot}92 - 36{\cdot}68)] \\ &= -\tfrac{1}{6}(0{\cdot}028) = -0{\cdot}005. \end{aligned}$$

$$\begin{aligned} B_1 &= \tfrac{1}{6}[0{\cdot}966(36{\cdot}55 - 36{\cdot}83 - 37{\cdot}09 + 36{\cdot}68) + 0{\cdot}707(36{\cdot}46 - 36{\cdot}72 \\ &\quad - 37{\cdot}30 + 36{\cdot}92) + 0{\cdot}259(36{\cdot}48 - 36{\cdot}62 - 37{\cdot}32 + 37{\cdot}17)] \\ &= -\tfrac{1}{6}(0{\cdot}667 + 0{\cdot}452 + 0{\cdot}075) = -0{\cdot}199. \end{aligned}$$

$$\begin{aligned} B_2 &= \tfrac{1}{6}[0{\cdot}866(36{\cdot}55 - 36{\cdot}48 - 36{\cdot}62 + 36{\cdot}83 + 37{\cdot}09 - 37{\cdot}32 \\ &\quad - 37{\cdot}17 + 36{\cdot}68)] \\ &= -\tfrac{1}{6}(0{\cdot}381) = -0{\cdot}064. \end{aligned}$$

$$\begin{aligned} B_3 &= \tfrac{1}{6}[0{\cdot}707(36{\cdot}55 - 36{\cdot}46 - 36{\cdot}48 + 36{\cdot}62 + 36{\cdot}72 - 36{\cdot}83 \\ &\quad - 37{\cdot}09 + 37{\cdot}30 + 37{\cdot}32 - 37{\cdot}17 - 36{\cdot}92 + 36{\cdot}68)] \\ &= \tfrac{1}{6}(0{\cdot}1696) = 0{\cdot}028. \end{aligned}$$

$\therefore$ Equation of curve is (to 2 places of decimals)

$$y = 36{\cdot}85 - 0{\cdot}36 \sin x + 0{\cdot}03 \sin 2x - 0{\cdot}01 \sin 3x - 0{\cdot}20 \cos x - 0{\cdot}06 \cos 2x + 0{\cdot}03 \cos 3x.$$

Table of values of y_1, y_2, . . . y_{12} for different values of x.

y	x	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$	$\sin 3x$	$\cos 3x$
y_1 = 36·55	15	0·2588	0·9659	0·5000 (30°)	0·8660 (30°)	0·7071 (45°)	0·7071 (45°)
y_2 = 36·46	45	0·7071	0·7071	1·0000 (90°)	0 (90°)	0·7071 (135°)	−0·7071 (135°)
y_3 = 36·48	75	0·9659	0·2588	0·5000 (150°)	−0·8660 (150°)	−0·7071 (225°)	−0·7071 (225°)
y_4 = 36·62	105	0·9659	−0·2588	−0·5000 (210°)	−0·8660 (210°)	−0·7071 (315°)	0·7071 (315°)
y_5 = 36·72	135	0·7071	−0·7071	−1·0000 (270°)	0 (270°)	0·7071 (45°)	0·7071 (45°)
y_6 = 36·83	165	0·2588	−0·9659	−0·5000 (330°)	0·8660 (330°)	0·7071 (135°)	−0·7071 (135°)
y_7 = 37·09	195	−0·2588	−0·9659	0·5000 (30°)	0·8660 (30°)	−0·7071 (225°)	−0·7071 (225°)
y_8 = 37·30	225	−0·7071	−0·7071	1·0000 (90°)	0 (90°)	−0·7071 (315°)	0·7071 (315°)
y_9 = 37·32	255	−0·9659	−0·2588	0·5000 (150°)	−0·8660 (150°)	0·7071 (45°)	0·7071 (45°)
y_{10} = 37·17	285	−0·9659	0·2588	−0·5000 (210°)	−0·8660 (210°)	0·7071 (135°)	−0·7071 (135°)
y_{11} = 36·92	315	−0·7071	0·7071	−1·0000 (270°)	0 (270°)	−0·7071 (225°)	−0·7071 (225°)
y_{12} = 36·68	345	−0·2588	0·9659	−0·5000 (330°)	0·8660 (330°)	−0·7071 (315°)	0·7071 (315°)

The following table compares the observed with the calculated values of y:—

y.	Calculated Value.	Observed Value.
y_1	36·54	36·55
y_2	36·46	36·46
y_3	36·51	36·48
y_4	36·61	36·62
y_5	36·72	36·72
y_6	36·84	36·83
y_7	37·10	37·09
y_8	37·30	37·30
y_9	37·33	37·32
y_{10}	37·15	37·17
y_{11}	36·92	36·92
y_{12}	36·70	36·68

The figures in the case of new-born babies yield the following equations:—

(1) For full-term infants (average of four cases)

$$y = 36{\cdot}382 + 0{\cdot}001 \sin x + 0{\cdot}223 \sin 2x + 0{\cdot}181 \cos x + 0{\cdot}069 \cos 2x$$

(2) For prematures

$$y = 37{\cdot}03 + 0{\cdot}27 \sin x + 0{\cdot}08 \sin 2x - 0{\cdot}04 \cos x - 0{\cdot}2 \cos 2x.$$

CHAPTER XXII.

MATHEMATICAL ANALYSIS APPLIED TO THE CO-ORDINATION OF EXPERIMENTAL RESULTS.—THE FINDING OF LAWS.

In order to be able to form an opinion of the nature of the process or processes responsible for any particular phenomenon under investigation, it is desirable to ascertain if there is any simple mathematical relationship existing between the dependent and independent variables—such as between the weight and height, weight and age, surface area and weight of a person; amount of chemical transformation and time; reaction-velocity and temperature; etc. Such a relationship expressed in the form of a mathematical formula, embodying the results found in the laboratory in a concise form, and which enables one to foretell with considerable certainty the quantitative results of any future observations of a similar nature, constitutes the *law* of the phenomenon in question, and forms a powerful weapon, not only for the detection of deviations between calculated and observed results, but also to enable one to form some idea of the causes of such deviations. It tells one, for instance, whether these deviations are due to experimental errors only or whether the discrepancies between the expected and observed values are greater than can be accounted for by errors due to faulty laboratory technique. In the former case such a formula leads the investigator to improve his laboratory methods with the object of minimising the magnitude of his errors; whilst in the latter the mathematical formula affords one an opportunity to ascertain what are the inherent factors responsible for such discrepancies, thereby helping to bring about further scientific discoveries.

The most classical as well as the most dramatic example of an important scientific discovery made in this way was the prediction by Adams and Leverrier, in 1846, of the mass, position and orbit of Neptune—as the result of the measured deviations of Uranus at different points in its calculated orbit—before its discovery and identification, a few months later, by

Galle, who directed his telescope to the place in the heavens indicated by the theoretical calculations.

It is the object of this chapter to show how a knowledge of the mathematical principles discussed in the foregoing chapters enables the investigator to find out whether the processes he has studied conform with any natural law, and if so, to express that law by means of a mathematical formula.

Theoretical and Empirical Formulæ.—(*a*) If there is any theoretical consideration to lead one to believe that the relationship between the dependent and independent variables is such as to follow some well-known law, then the problem is fairly easy. All one has to do is to compare the observed results with those that one would expect to find by using the formula, and see whether, allowing for errors of observation, the observed and calculated results agree. If they do, then there is a great probability (though by no means an absolute certainty) that the assumed formula is the correct one. A formula so established constitutes a *theoretical formula*.

The following examples will make this clear:—

(1) *Problem in Physiology of Growth.*—Robertson, as the result of certain theoretical considerations (see "*Child Physiology*," Chapter XVIII.), concluded that the growth in weight of a fœtus, as well as of a young infant, is an autocatalytic phenomenon for which he derived the following equation:—

$$K = \frac{1}{t - 1{\cdot}66} \log_{10} \frac{x}{341{\cdot}5 - x}$$

where x = weight in ozs. of infant (or fœtus) at t months from birth (in the case of the infant t is $+$ve, and of the fœtus $-$ve), and K = growth constant.

The following table gives the corresponding values of t and x for fœtuses and infants at various ages. Find whether the formula is true. Also calculate the theoretical weight of an infant at the age of eight months, using the mean value of K thus found.

t	$-0{\cdot}75$	$-0{\cdot}42$	$-0{\cdot}08$	0 (birth)	$+0{\cdot}25$	$+0{\cdot}58$	$+0{\cdot}92$
x	111	117	127	127	137	145	146

If the formula represents the true relationship between t and x, then by substituting in it corresponding pairs of values of t

and x, the values of K thus found should be practically identical in each case (but for errors of observation). If we do this we get the results given in the table below.

The agreement between the various values of K is seen to be very good and the formula probably, therefore, represents the true relationship between t and x. (But see note on p. 90, Chapter VII.)

To find the weight of an infant eight months old, put $t = 8$; we then have

$$\text{mean K} = 0\cdot136 = \frac{1}{8 - 1\cdot66} \log_{10} \frac{x}{341\cdot5 - x}$$

$$= \frac{1}{6\cdot34} \log_{10} \frac{x}{341\cdot5 - x}$$

$$\therefore \quad 0\cdot136 \times 6\cdot34, \textit{ i.e. } 0\cdot862, = \log_{10} \frac{x}{341\cdot5 - x}$$

But 0·862 is the logarithm of 7·278.

$$\therefore \quad 7\cdot278 = \frac{x}{341\cdot5 - x}$$

This gives x = 300 ozs., *i.e.* the theoretical weight of a normal infant eight months old is 300 ozs. (This also is the observed weight.)

t.	x.	$\frac{1}{t - 1\cdot66} \log_{10} \frac{x}{341\cdot5 - x} =$ K.
$-0\cdot75$	111	$-\frac{1}{2\cdot41} \log \frac{111}{230\cdot5} = 0\cdot132$
$0\cdot42$	117	$-\frac{1}{2\cdot08} \log \frac{117}{224\cdot5} = 0\cdot136$
$-0\cdot08$	127	$-\frac{1}{1\cdot74} \log \frac{127}{214\cdot5} = 0\cdot131$
0	127	$-\frac{1}{1\cdot66} \log \frac{127}{214\cdot5} = 0\cdot137$
$+0\cdot25$	137	$-\frac{1}{1\cdot41} \log \frac{137}{204\cdot5} = 0\cdot123$
$+0\cdot58$	145	$-\frac{1}{1\cdot08} \log \frac{145}{196\cdot5} = 0\cdot122$
$+0\cdot92$	146	$-\frac{1}{0\cdot74} \log \frac{146}{195\cdot5} = 0\cdot171$
		Mean = 0·136

(2) *Problem in Psychology.*—There is some experimental evidence of a chemical nature to show that mental processes are of the nature of an autocatalytic chemical reaction. Investigate the following figures obtained by Ebbinghaus in a certain memory test to see whether they agree with expectation, assuming the particular autocatalytic equation in this case to be

$$\log \frac{x}{43\cdot6 - x} = 0\cdot001468t - 0\cdot526 \text{ (Robertson)}$$

x	7 (1)	16 (30)	24 (44)	36 (55)
t	2·8	192	422·4	792

x = number of meaningless syllables repeated a number of times (shown in brackets) at the rate of 0·4 second per syllable,

t = number of seconds required to fix that number of syllables upon the memory.

If we substitute the various values of t in the equation

$$\log \frac{x}{43\cdot6 - x} = 0\cdot001468t - 0\cdot526,$$

we get as follows:—

When $t = 2\cdot8$, $\log \dfrac{x}{43\cdot6 - x} = 0\cdot001468 \times 2\cdot8 - 0\cdot526$

$$= -0\cdot5218896,$$

or $$\log \frac{43\cdot6 - x}{x} = 0\cdot5218896 = \log 3\cdot326,$$

i.e. $$\frac{43\cdot6 - x}{x} = 3\cdot326,$$

whence $$x = 10\cdot08.$$

Similarly, when $t = 192$, $\log \dfrac{x}{43\cdot6 - x} = -0\cdot244$,

$$\therefore \quad \frac{43\cdot6 - x}{x} = 1\cdot754.$$

$$\therefore \quad x = \frac{43\cdot6}{2\cdot754} = 15\cdot8.$$

Working in the same way with the other values of t, we get

when $t = 422{\cdot}4,\ x = 24{\cdot}2$

" $t = 792{\cdot}0,\ x = 35{\cdot}4.$

Hence, except for the first value of t, the agreement between the observed and calculated values of x is excellent.

(*b*) If, however, there is no theoretical basis to guide one in the selection of the formula, then the procedure is as follows:—

A number of pairs of values of the two variables is taken and these are plotted in the form of a graph. If the result seems to be a straight line (within the limits of errors of observation) then one knows that the required formula is one of the first degree in x and y and is of the form $y = mx+b$ (which is the general equation of a straight line (p. 112)). What remains, therefore, is to find the values of the constants m and b. This can be done very easily either by simple algebra or by inspection, since m is the slope, and b is the y intercept of the line.

EXAMPLES.

(1) The following table records the solubility of $NaNO_3$ in water at various temperatures (S = weight in grammes of $NaNO_3$ dissolved in 100 grammes of water; t = temperature of the water in degrees Centigrade):—

S	68·4	72·9	87·5	102
t	−6	0	20	40

By plotting the graph we find it to be a straight line (fig. 128). We therefore write down the empirical formula as

$$S = mt+b.$$

The value of b is found at once, because when $t = 0$, S becomes $m \times 0 + b = b$.

But when $t = 0, \quad S = 72{\cdot}9.$

$$\therefore \quad b = 72{\cdot}9.$$

Hence the equation becomes

$$S = mt + 72{\cdot}9.$$

To find the value of m take any other pair of values of S and t from the table and put them into the equation. For example, take S = 68·4 and $t = -6$. We then get

$$68{\cdot}4 = -6m + 72{\cdot}9,$$

giving $m = 0{\cdot}75$.

Hence the equation becomes

$$S = 0{\cdot}75t + 72{\cdot}9.$$

This equation, however, is only approximately correct, because if we were to take any other pairs of values of S and t, we would get slightly different values of m.

Thus when S = 87·5 and t = 20, the equation becomes 87·5 = $20m + 72{\cdot}9$,

whence $$m = 0{\cdot}73.$$

Similarly when S = 102 and t = 40, the value of m becomes 0·7275.

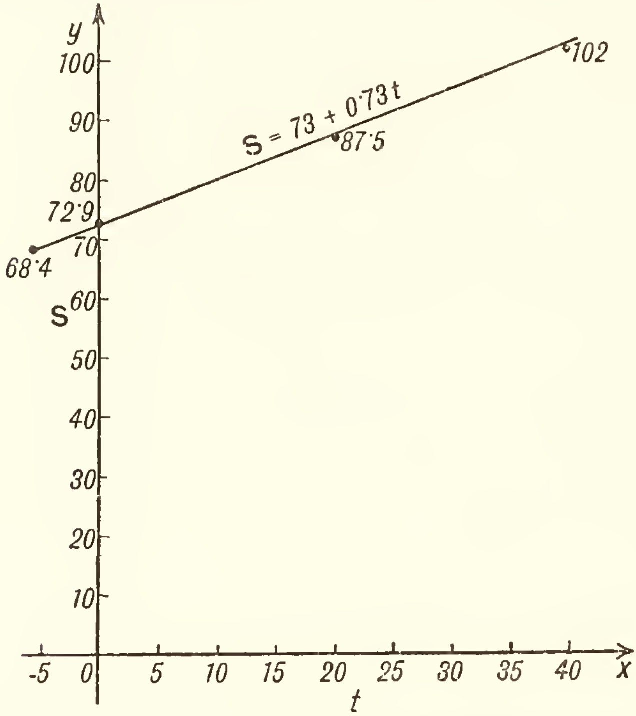


FIG. 128.—Graph of Solubility of $NaNO_3$ in Water at Various Temperatures.

It is possible by utilising the *method of least squares* (see next Chapter, p. 423) to obtain a more accurate equation. When this is done the final result becomes

$$S = 0{\cdot}73t + 72{\cdot}9.$$

(Direct measurement of the slope of the line gives $m = 0{\cdot}73$.)

Having obtained this formula it is obvious that the solubility of $NaNO_3$ at any other temperature can be easily predicted by calculation or by interpolation or extrapolation.

Thus at 10° C.	S = 0·73 × 10 + 72·9 = 80·2,
,, 35° C.	S = 0·73 × 35 + 72·9 = 98·5,
,, 50° C.	S = 0·73 × 50 + 72·9 = 109·4,

and so on.

The student might work the next example as an exercise.

(2) In the case of KBr, the following values of S were found for the stated values of t:—

S	53·4	64·6	74·6	84·7	93·5
t	0	20	40	60	80

Find the law connecting S and t and use it for calculating S at 10°, 55°, and 100° C.

[*Answer*, $S = 54{\cdot}2 + 0{\cdot}5t$; $S_{10} = 59{\cdot}2$; $S_{55} = 81{\cdot}7$; $S_{100} = 104{\cdot}2$.]

It is when the resulting graph is not a straight line that the difficulty of finding the empirical formula from the shape of the curve becomes very considerable, since it is, as a rule, impossible to tell by mere inspection whether the portion of the curve plotted belongs to a parabola, a hyperbola, a logarithmic curve, etc. The empirical formula must in such cases be obtained by trial. Generally one has some sort of an idea as to what kind of formula one may expect, and in order to test the correctness of one's expectations one can, by means of certain artifices, attempt to convert the curve into a straight-line graph. If the attempt is successful then the presumption is that the formula assumed is the correct one.

The method is best illustrated by means of a few worked-out examples.

EXAMPLES.

(1) Sjöquist studied the course of pepsin digestion by measuring the electrical conductivity (x) of the protein solution. He found the following values of x at the various times t (in hours):—

x	0	10·5	16·41	19·93	22·68	24·00	27·04	30·36	33·68
t	0	2	4	6	8	9	12	16	20

Find the law connecting x and t, and state to what family of curves the corresponding graph belongs.

The graph (fig. 129) looks like a portion of a parabola. To see whether it is such, let the law connecting x and t be put into the form

$$t = kx^2, \quad \text{or} \quad \sqrt{t} = Kx \text{ (where K is another constant} = \sqrt{k}\text{)}.$$

If this form is correct, then x plotted against $\sqrt{t}$ should give a straight line.

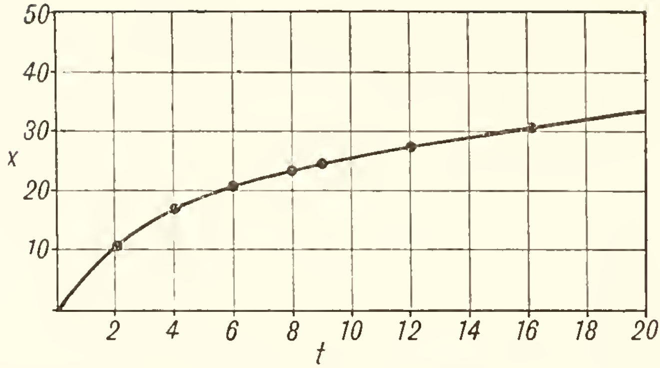


FIG. 129.—Graph of Peptic Digestion. (Result of plotting x against t.)

The plotting table will be:

x	0	10·5	16·41	19·93	22·68	24·00	27·04	30·36	33·68
$\sqrt{t}$	0	1·41	2·0	2·45	2·83	3·0	3·46	4·0	4·47

As will be seen from fig. 130, these values give a straight line passing through the origin.

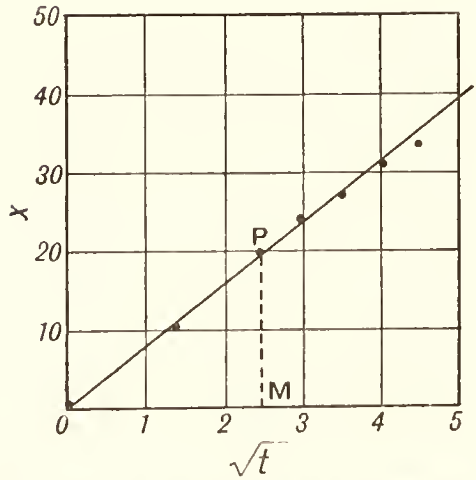


FIG. 130.—Graph of Peptic Digestion. (Result of plotting x against $\sqrt{t}$.)

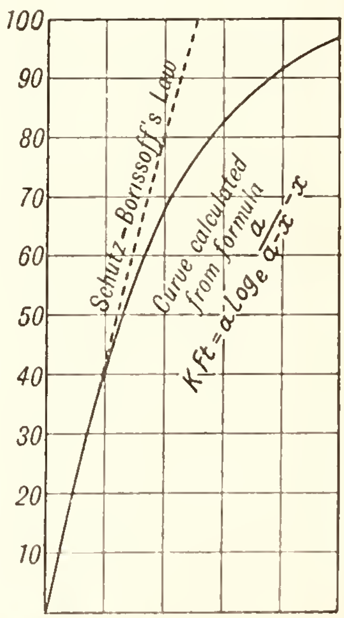


FIG. 131.—Linear and Logarithmic Graphs of the Schütz-Borissoff Law.

Therefore the law is $\sqrt{t} = Kx$, and the graph is a parabola. In other words, the course of pepsin digestion follows the Schütz-Borissoff law.

The value of K is seen from the graph to be about 1/8

$$\left(e.g.\ \frac{OM}{PM} = \frac{2{\cdot}45}{19{\cdot}93} = 0{\cdot}12\right).$$

$$\therefore \quad \text{The law is} \qquad \sqrt{t} = 0{\cdot}12x \quad \text{or} \quad x = \frac{\sqrt{t}}{0{\cdot}12}.$$

This example furnishes a very good illustration of the danger of using a formula for extrapolation purposes. Thus Bayliss has shown that the Schütz-Borissoff law as expressed by the relation $x = K\sqrt{t}$ only holds good for a certain stage of digestion, and therefore whilst it is safe to use this formula for purposes of *interpolation* within certain values of x or t, its adoption for the purpose of *extrapolation* outside those limits would give totally erroneous results.

Thus we have seen on p. 333 that the true equation for peptic digestion is $KFt = a \log_e \frac{a}{a-x} - x$, but that for small values of x the relation $x = K\sqrt{aFt}$ holds good. Fig. 131 shows the course of the reaction according to the logarithmic equation as well as according to the Schütz-Borissoff law. It is seen that up to a certain point the two graphs are practically coincident. But if extrapolation of $x = K\sqrt{aFt}$ were tried from the moment where the straight line is shown dotted, the results would be totally misleading.

(2) The following values of x and y are assumed to be connected by an equation of the form $y = a + bx^2$:—

x	0	1	2	3	4	5	6	7	8
y	2	2·05	2·2	2·45	2·8	3·25	3·8	4·45	5·2

Test the correctness of this assumption and find the values of the constants a and b.

Plotting y and x gives the curve shown in fig. 132, but it is obvious that mere inspection altogether fails to identify the type of curve to which the plotted portion belongs. From the formula $y = a + bx^2$ we would expect it to be a portion of a parabola. This it might well be, but it might from its appearance equally well be a portion of a hyperbola or of an exponential curve. If, however, instead of plotting y and x, as we have done, we plot y and x^2, then, if the resulting graph is a straight

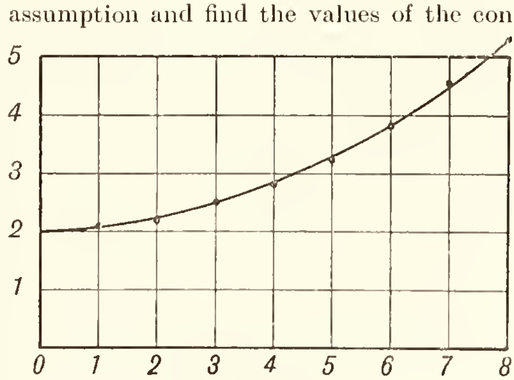


FIG. 132.

line we know that $y = a + bx^2$ is the correct formula connecting the given values of x and y.

For, putting $x^2 = X$, the equation becomes $y = a + bX$, which represents a straight line.

If we look at the graph (fig. 133) resulting from plotting y against x^2, we see that it is a straight line, and therefore the assumption is correct.

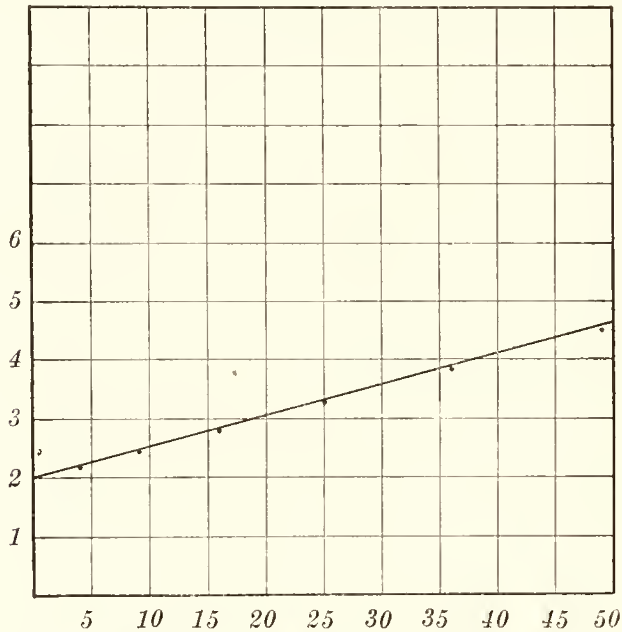


FIG. 133.—Modification of Graph in fig. 132 obtained by plotting y against x^2.

To find the values of a and b, we proceed exactly as in Example (1) on p. 353.

When $x = 0, \quad y = a;$

$\therefore \quad a = 2.$

To find b, take any pair of values of X (or x^2) and y, and put them into the equation $y = a + bx^2$.

$$\text{Thus when } y = 2{\cdot}2, \quad x^2 = 4;$$

$$\therefore \quad 2{\cdot}2 = 2 + 4b;$$

$$\therefore \quad b = \frac{0{\cdot}2}{4} = 0{\cdot}05.$$

$\therefore$ Equation is $\quad y = 2 + 0{\cdot}05x^2.$

The value of the slope b can also be read off directly from the straight line graph, when it is seen to be equal to 0·05.

Note.—Whenever we have reason to expect that $\frac{dy}{dx}$ is proportional to x, the formula $y = a + bx^2$ must be tried $\left(\text{since } \frac{dy}{dx} = 2bx\right)$.

(3) The following values of x and y are believed to be related by an equation of the form $y = Ae^{bx}$ (*i.e.* the phenomenon in question is supposed

to be an example of the compound interest law). Examine if this is so and then evaluate A and b.

x	120	110	100	90	80	70	60
y	0·0051	0·0059	0·0071	0·0085	0·0102	0·0124	0·0148

If we were to plot x and y we would get a portion of a curve (fig. 134)

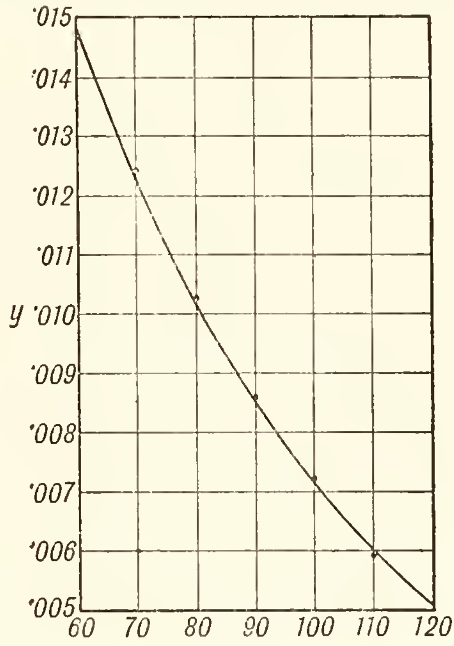


Fig. 134.

which might be exponential, but might equally be a portion of a parabola or of some other curve. But by taking logarithms of both sides we get:

$$\log_{10} y = \log_{10} A + bx \log_{10} e = \log_{10} A + 0{\cdot}4343bx,$$

which is an equation of the first degree in x and y. If, therefore, the given values of x and y are related by an equation of the form $y = Ae^{bx}$, then, by plotting x and $\log y$, we should get a straight line whose y intercept is $\log_{10} A$ and whose slope is $0{\cdot}4343b$.

The plotting table will be:

x	120	110	100	90	80	70	60
$\log_{10} y$	$\bar{3}{\cdot}7076$	$\bar{3}{\cdot}7709$	$\bar{3}{\cdot}8513$	$\bar{3}{\cdot}9294$	$\bar{2}{\cdot}0086$	$\bar{2}{\cdot}0934$	$\bar{2}{\cdot}1703$

and the graph so plotted is seen to be a straight line (fig. 135).

Hence the assumption that $y = Ae^{bx}$ is true. In order to find the values of A and b, we take any two pairs of values on the graph;

e.g. when $x = 80$, $\log y = \bar{2}\cdot0086$,

and when $x = 100$, $\log y = \bar{3}\cdot8513$.

Hence

$$\bar{2}\cdot0086 = \log_{10} A + 0\cdot4343 \times 80b \quad . \quad . \quad . \quad (1)$$

and

$$\bar{3}\cdot8513 = \log_{10} A + 0\cdot4343 \times 100b \quad . \quad . \quad . \quad (2)$$

By subtraction, $\bar{1}\cdot8427 = 0\cdot4343 \times 20b$.

i.e. $-0\cdot1573 = 8\cdot686b$.

$$\therefore \quad b = -\frac{1573}{86860} = -0\cdot018.$$

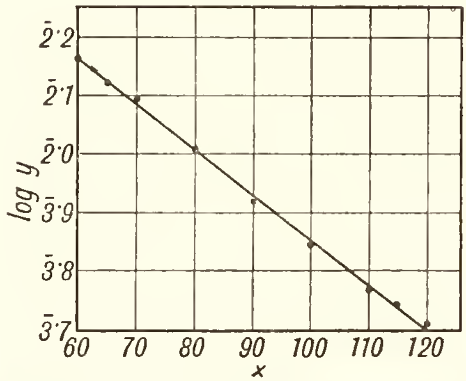


Fig. 135.—Modification of Graph in fig. 134, obtained by plotting $\log_{10} y$ against x.

Substituting for b in equation (2) we get

$$\bar{3}\cdot8513 = \log A - 0\cdot4343 \times 1\cdot8$$
$$= \log A - 0\cdot7817.$$
$$\therefore \quad \log A = \bar{3}\cdot8513 + 0\cdot7817$$
$$= \bar{2}\cdot6330.$$
$$\therefore \quad A = 0\cdot043.$$

$\therefore$ Final equation is

$$y = 0\cdot043e^{-0\cdot018x}$$

The value of b could be ascertained by mere inspection:

$0\cdot4343b = \tan\theta$, where θ is the angle made by the straight line with the x axis.

$\tan\theta$ is obviously $-\dfrac{0\cdot47}{60} = -0\cdot0078.$

$$\therefore \quad b = -\frac{0\cdot0078}{0\cdot4343} = -0\cdot018.$$

(4) The activity (in arbitrary units) of a certain volume of radium emanation was as follows:—

Time (hours)	0	20·8	187·6	354·9	521·9	786·9
Activity .	100	85·7	24	6·9	1·5	0·19

Find if the relationship between t and A corresponds to the compound interest law, and calculate the radioactive constant.

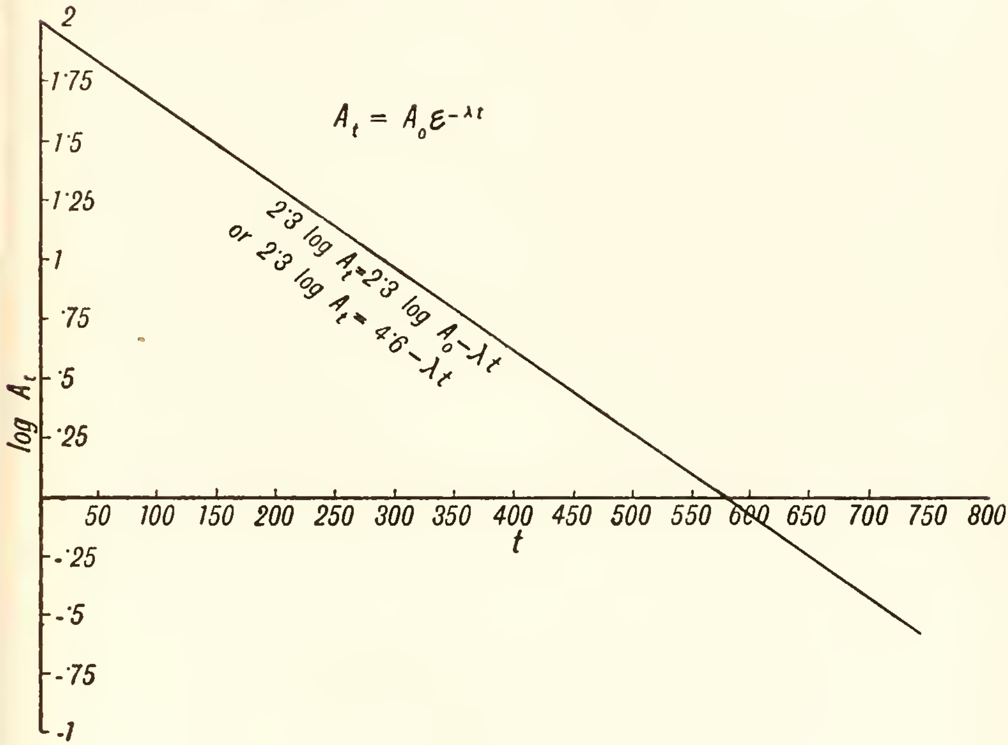


FIG. 136.

If this is an example of the compound interest law, then

$$A_t = A_0 e^{-\lambda t}$$

(where A_t = activity at time t, A_0 = original activity = 100, λ = radioactive constant).

$$\therefore \quad \log A_t = \log A_0 - \lambda t \log e = 2 - 0{\cdot}4343\lambda t$$
$$(\text{or } 2{\cdot}3 \log A_t = 4{\cdot}6 - \lambda t).$$

Hence by plotting log A_t against t (in accordance with the next plotting table) we should obtain a straight line, as indeed is the case (fig. 136):

t . .	0	20·8	187·6	354·9	521·9	786·9
log A_t .	2	1·933	1·380	0·839	0·176	−0·721

Hence the formula $A_t = A_0 e^{-\lambda t}$ is correct.

The value of λ is easily found from the graph, since $-0{\cdot}4343\lambda$ is the value of the tangent of the angle which the line makes with the x axis, which is seen to be $-2/580$, so that $\lambda = 2 \times 2{\cdot}3/580 = 0{\cdot}0079$.

Hence the equation representing the activity of the particular emanation is $A_t = 100e^{-0{\cdot}0079t}$.

(5) The area of a wound was determined every four days by applying a sheet of transparent cellophane and making drawings on it of the edges of the wound. The areas of these drawings were then measured by the planimeter, and the following were the corrected results obtained:—

No. of days . .	0	4	8	12	16	20	24	28	32	36
Area in sq. cm. (A)	107	88	74·2	61·8	51	41·6	33·6	26·9	21·3	16·8

Show that cicatrisation follows the compound interest law and find the appropriate equation. After how many days should the size of the wound be 1 square centimetre?

By plotting *log area* against *time* we get the following table (to two places of decimals):—

t (days) .	0	4	8	12	16	20	24	28	32	36
log A .	2·03	1·94	1·87	1·79	1·71	1·62	1·53	1·43	1·33	1·23

The resulting graph is a straight line. Let its equation be

$$\log A_t = 2{\cdot}03 - 0{\cdot}4343kt.$$

$$\therefore \quad -0{\cdot}4343k \text{ is its slope} = -\left(\frac{2{\cdot}03 - 1{\cdot}23}{36}\right) = -0{\cdot}022,$$

$$\therefore \quad k = 0{\cdot}051.$$

Hence the appropriate equation is

$$A_t = 107e^{-0{\cdot}051t} \quad \text{or} \quad \log A_t = 2{\cdot}03 - 0{\cdot}022t.$$

When $A_t = 1$ sq. cm., $\log A_t$ is 0, $\therefore$ $2{\cdot}03 = 0{\cdot}022t$, whence $t = 92$ days.

Note.—Carrel, Hartmann, Lecomte du Nouy, and others (*J. Exp. Med.*, xxiv, 1916, and xxvii, 1918) have shown that rate of cicatrisation of a wound generally follows the compound interest law. In this way it is possible to calculate (in the case of any aseptic wound) the size it will be at any given date and to foretell when cicatrisation will be complete. Marked deviation from the calculated graph shows that infection has occurred. Also the action of different dressings and antiseptics can be studied.

(6) A. G. M'Kendrick and M. Kesava Pai (see p. 334, Example (6)), working with *Bacillus coli*, found that when starting with 2850 bacilli they

obtained the following numbers y after growing for various intervals of time t:—

t (in hours) .	0	1	2	3	4
y . .	2850	17,500	105,000	625,000	2,250,000

They further found that when the number reached 100,000,000 growth ceased.

Find whether the figures agree with the equation of growth given in Example (6) on p. 334, viz. $\frac{dy}{dt} = by(a-y)$, and evaluate the constants.

Also calculate the *period of a generation*, *i.e.* the time it takes for a bacillus to double itself.

From
$$\frac{dy}{dt} = by(a-y)$$
we get
$$\frac{1}{y}\frac{dy}{dt} = b(a-y).$$
But
$$\frac{1}{y} = \frac{d\log_e y}{dy},$$
$$\therefore \quad \frac{d\log_e y}{dy}\cdot\frac{dy}{dt} = b(a-y),$$
i.e.
$$\frac{d\log_e y}{dt} = b(a-y).$$

In other words, the slope of the curve obtained by plotting $\log_e y$ against t is $b(a-y)$.

Now, when $t = 0$, $y = 2850 = y_0$, and when growth has ceased $y = a = 100{,}000{,}000$ (see Example (6), p. 335).

$\therefore$ Slope at commencement of curve, which is $\frac{d\log_e y_0}{dt}$, $= b(a-y_0) = ba$ (since y_0 is small compared with a), or $\frac{d\log_{10} y_0}{dt} = 0{\cdot}4343ab$.

Hence, on plotting $\log y$ against t, the slope at the commencement of the curve $= 0{\cdot}4343ba$. But a is known, and therefore b can be found.

The new plotting table is:

t	0	1	2	3	4
$\log_{10} y$	3·455	4·243	5·021	5·796	6·352

The first portion of the graph will be found to be practically a straight line whose slope $= \tan^{-1} 0{\cdot}8$.

But $a = 10^8$, and $ab = 0{\cdot}8 \times 2{\cdot}3 = 1{\cdot}84$.

$$\therefore \quad b = 1{\cdot}84 \times 10^{-8}.$$

To find the *period of generation*, we notice that ab is the rate of change of $\log_{10} y$ per unit of time (*i.e.* per hour).

In one hour $\log_{10} y$ has changed from 3·455 to 4·243, *i.e.* by approximately 0·80.

But when y has doubled itself, $\log_{10} y$ has changed by $\log_{10} 2$, *i.e.* by 0·301.

$$\therefore \quad \text{Period of a generation} = \frac{0·301}{0·8} \text{ hr.} = 22·5 \text{ mins.}$$

(7) The following values of x and y have been observed. Find a formula connecting them.

x	0	1	2	3	4	5	6	7
y	0	0·7485	0·5988	0·5614	0·5444	0·5347	0·5284	0·5241

Here y decreases with increase of x; we therefore try the formula $y = \frac{ax}{x-b}$, or $x = \frac{ax}{y} + b$ and plot x against $\frac{x}{y}$ as follows:—

x	0	1	2	3	4	5	6	7
$\frac{x}{y}$	0	1·336	3·340	5·343	7·348	9·351	11·35	13·56

The resulting graph is a straight line.

∴ The assumption that the observed values might be connected by the equation $x = a\frac{x}{y} + b$ is true.

To find a and b we proceed exactly as in the other cases of straight line laws (Example (1), p. 353), and we get $a = 0·5$ and $b = 0·33$.

$$\therefore \quad \text{Equation is} \qquad x = 0·5\frac{x}{y} + 0·33,$$

or

$$xy = 0·5x + 0·33y.$$

When other formulæ fail then one uses the equation

$$y = a + bx + cx^2 + dx^3 + ex^4 + \ . \ . \ .$$

because it is known that such an equation will satisfy any values of x and y provided we take a sufficient number of terms. Thus, if we stop at the first power of x we get $y = a + bx$, which, of course, represents a straight line; $y = a + bx + cx^2$ will correspond to a parabola; whilst, as we have seen, any function, whether logarithmic, exponential, trigonometric, etc., can be expanded into a series of ascending powers of x like the above.

In order to evaluate the coefficients a, b, c, d . . . we make use of Maclaurin's theorem (p. 228), according to which

$$a = f(0); \quad b = f'(0); \quad c = \frac{f''(0)}{2\,!}; \quad d = \frac{f'''(0)}{3\,!}; \ldots$$

Hence a is simply the numerical value of the intercept of the y axis cut off by the original, or primitive, curve. To determine the numerical values of $b, c, d,$ . . . we must plot the curves $y' = f'(x)$, $y'' = f''(x)$, etc. $y' = f'(x)$, *i.e.* the first slope curve or first derivative curve, can be plotted by drawing tangents to the original curve at various points and calculating the slopes at those points in the manner explained on p. 165 *et seq.* and in Examples (6) and (7) on p. 379. The values of y' so obtained are plotted against corresponding values of x, and we thus get the first derivative curve. By repeating this process we get the second derivative curve $y'' = f''(x)$. The process is repeated until we get a graph which does not differ perceptibly from a straight line.

If, now, we measure the intercept of the y axis cut off by each of these derivative curves we obtain the values $\frac{f'(0)}{1\,!}, \frac{f''(0)}{2\,!}, \frac{f'''(0)}{3\,!}$, etc., and, hence, we get the numerical values of b, c, d, etc.

Note.—In practice one does not often employ the method just described for obtaining the various derived curves since, although theoretically perfect, practically it is very inaccurate (see p. 372). For the most exact methods of plotting these curves the reader is referred to more advanced books on practical mathematics. By choosing appropriate scales for x and y, however, the curve can be drawn in such a way as to make the foregoing methods applicable with reasonable accuracy.

EXAMPLES.

(1) The following have been found to be respective values of x and y. Find the law connecting them.

x	0	1	2	3	4	5	6	7	8
y	2	1·85	1·8	1·85	2	2·25	2·6	3·05	3·6

Let the equation of the plotted curve (fig. 137) be $y = a + bx + cx^2 + \ldots$

By drawing tangents at the points where $x = 1, 2, 3,$ . . . we obtain the numerical values of the slopes as follows:—

x	1	2	3	4	5	6
y'	−0·1	0	0·1	0·2	0·3	0·4

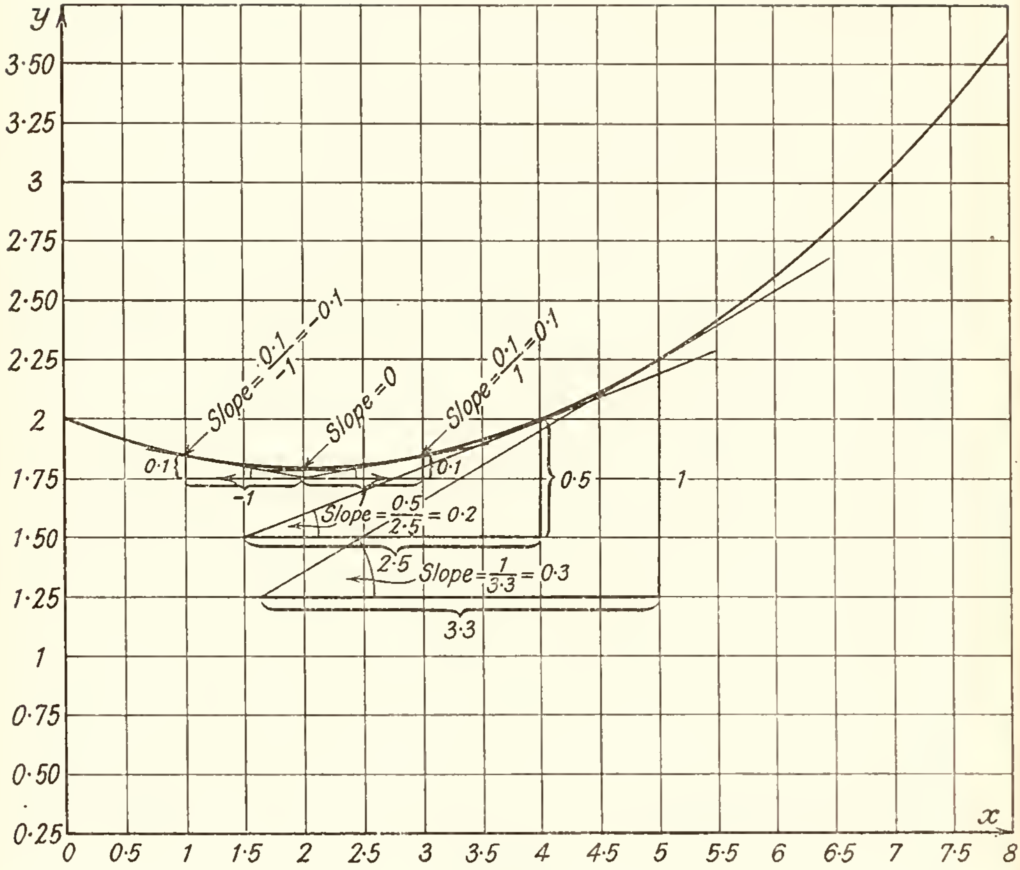


Fig. 137.

When these points are plotted they are found to be on a straight line (fig. 138) (which is, of course, $y' = f'x = b + 2cx$) whose slope is 0·1 and whose intercept on the y axis is $-0{\cdot}2$.

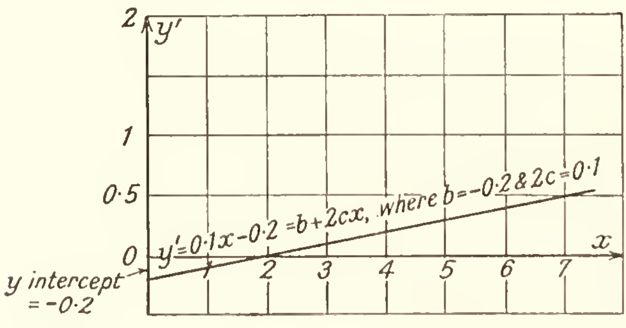


Fig. 138.

$\therefore$ Equation of line is $y' = 0{\cdot}1x - 0{\cdot}2$.

$\therefore$ $b = -0{\cdot}2$ and $c = 0{\cdot}05\left(=\frac{f''(0)}{2!}\right)$.

But $a = f(0) = 2$.

$\therefore$ The law connecting the two variables, or the equation of the primitive curve, is $y = 2 - 0{\cdot}2x + 0{\cdot}05x^2$. The curve is therefore a parabola (fig. 137).

(2) The following values have been found for x and y. Find the law connecting them.

x	-3	-2	-1	0	1	2	3	4	5
y	0	24	30	24	12	0	-6	0	24

Plot the curve (fig. 139) by making one unit length on the x axis equal to 10 units on the y axis. It will then be found that the slopes can be

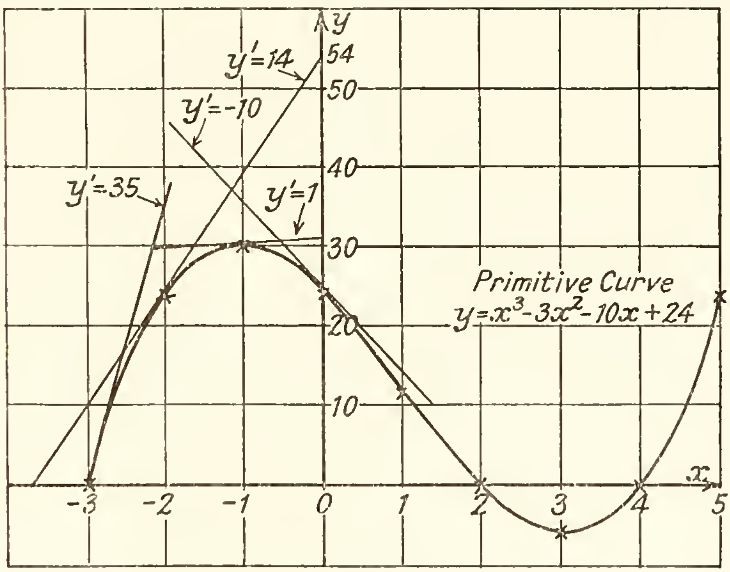


Fig. 139.

determined with fair accuracy at various points. The following will be found to be the values of y' for the stated values of x:—

x	-3	-2	-1	0	2	3	4
y'	35	14	1	-10	-10	-1	14

This will plot into what looks like a parabola (fig. 140), crossing the y axis

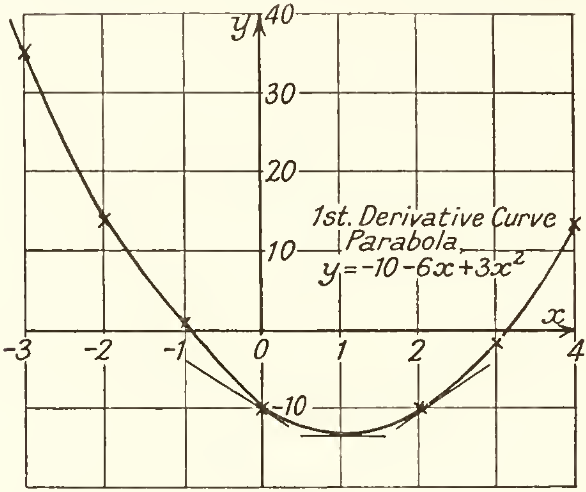


FIG. 140.

at $y = -10$, and calculating the slopes at various points on this first derivative curve, one gets the following values of y'' and x:—

x	-1	0	1	2
y''	-12	-6	0	6

The graph of this (fig. 141) is $y'' = 6x - 6$,

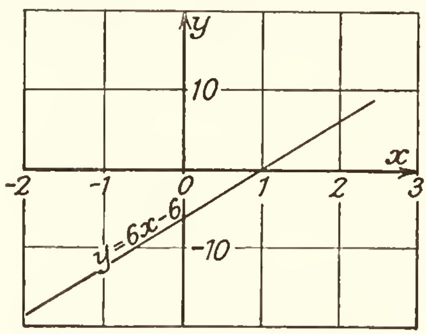


FIG. 141.

$$\therefore \quad a = f(0) = 24; \quad b = f'(0) = -10;$$

$$c = \frac{f''(0)}{2!} = \frac{6}{2} = -3; \quad d = \frac{f'''(0)}{3!} = \frac{6}{6} = 1$$

$\therefore$ Law connecting x and y is

$$y = x^3 - 3x^2 - 10x + 24.$$

Summary.—If there is no theoretical basis to guide us and the smooth graph plotted from the pairs of values of x and y is not a straight line, then one tries any of the following methods:—

(1) Plot y against x^2 or against $\sqrt{x}$. If the result is a straight line then the equation is of the form $y = a + bx^2$ or $y = a + b\sqrt{x}$.

(2) Plot y against $\log x$ or $\log y$ against x. If the result is a straight line then the equation is of the form $y = Ae^{bx}$.

(3) Plot $\log y$ against $\log x$. If the result is a straight line the equation is of the form $y = ax^n$.

(4) If $y = K\frac{y}{x}$, try the formula $y = \frac{ax}{x-b}$.

(5) If the functions seem to be periodic, try Fourier's series:

$$y = a_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \cos 2x + \ . \ . \ .$$

(see p. 339 *et seq.*).

(6) If all the foregoing formulæ fail, use the general equation $y = a + bx + cx^2 + dx^3 + \ . \ . \ .$ and evaluate the constants in the manner described on p. 365. It may, for instance, be found that $\frac{dy}{dx} = Kx$, when we know that the equation is of the form $y = a + bx^2$ (Schütz-Borissoff law), or it may be found that $\frac{dy}{dx} = Ky$, when the equation is an example of the compound interest law, $y = Ae^{bx}$, etc.

Determination of the Order of a Chemical Reaction.—Another most useful application of the method of graphical analysis is for the purpose of determining the order of a chemical reaction —by which is meant the number of "molecules" taking part in the reaction. For example, the splitting of hydrogen peroxide into water and oxygen by the hæmase of defibrinated blood is unimolecular, or a reaction of the first order, thus $H_2O_2 = H_2O + O$ (see p. 263).

The saponification of an ester is bimolecular, or of the second order, thus:

$$\underset{\text{Ethyl acetate.}}{CH_3COOC_2H_5} + NaOH = \underset{\text{Sodium acetate.}}{CH_3COONa} + \underset{\text{Ethyl alcohol.}}{C_2H_5OH}.$$

In many cases in biochemistry it is not easy to represent the reaction by means of an exact equation, and hence it is difficult or impossible to determine the order of the reaction by mere inspection. Moreover, even when it is possible to represent the reaction by means of an equation, it is not always possible to

say with certainty whether the reaction is uni-, bi- or multimolecular. For instance, in the case of the decomposition of H_2O_2 by hæmase, although we know that one molecule of H_2O_2 gives rise to one molecule of H_2O, it is not possible, without the aid of mathematics, to say whether the order is first or second, *i.e.* whether the equation is

$$H_2O_2 = H_2O + O,$$

or

$$2H_2O_2 = 2H_2O + O_2.$$

(1) One method of determining the order of a reaction we have already dealt with in Chapter XV. It consists in calculating the velocity constant K by means of the formulæ for unimolecular, bimolecular, etc., reactions. The reaction is of that order for which the formula gives reasonably constant values of K.

The following are *graphical methods* of determining the order of a reaction:—

(2) We have seen that $\frac{dx}{dt} = K(a-x)^n$, where n = order of the reaction.

If we put $(a-x) = C$ (*i.e.* concentration of the original substance at any instant t),

then
$$\frac{dx}{dt} = -\frac{dC}{dt}$$

and
$$(a-x)^n = C^n$$

$\therefore$ Equation becomes

$$-\frac{dC}{dt} = KC^n$$

$$\therefore \quad \frac{dC}{C^n} = -K\,dt$$

$$\therefore \quad \int\frac{dC}{C^n} = -Kt + A \text{ (A = integration constant).}$$

Hence,

(i) **In the case of unimolecular reactions** ($n = 1$),

$$\int\frac{dC}{C} = -Kt + A,$$

i.e.
$$\log_e \mathbf{C} = -\mathbf{Kt} + \mathbf{A}.$$

(ii) **In the case of bimolecular reactions** ($n = 2$),

$$\int\frac{dC}{C^2} = -Kt + A,$$

i.e.
$$-\frac{1}{C} = -Kt + A,$$

or
$$\frac{1}{C} = Kt - A.$$

(iii) **In the case of termolecular reactions** ($n = 3$),

$$\int C^{-3}dC = -Kt + A,$$

or
$$-\frac{1}{2C^2} = -Kt + A,$$

or
$$\frac{1}{2C^2} = Kt - A.$$

(iv) **In the case of n-molecular reactions,**

$$\int C^{-n}dC = -Kt + A,$$

or
$$\frac{1}{(1-n)C^{n-1}} = -Kt + A.$$

Hence, we get the following *rule*:—

To find the order of a reaction we have to ascertain by trial which of the following expressions gives a straight line when plotted against time as abscissa.

(i) $\log_e C$:—**Reaction is unimolecular.**

(ii) $\frac{1}{C}$:—**Reaction is bimolecular.**

(iii) $\frac{1}{C^2}$:—**Reaction is termolecular.**

(iv) $\frac{1}{C^{n-1}}$:—**Reaction is n-molecular.**

This method, therefore, like the last, involves a certain number of trials in order to arrive at the solution. There is, however, another graphical method, in which by drawing the graph one can, by means of a mathematical formula, arrive at the solution at once.

(3) **The Differential Method** (see Examples, p. 379 *et seq.*).—Draw on a large scale a graph representing the change of concentration (x) with time (t). Then the reaction velocity at any point can be determined by measuring the angle which the tangent at that point makes with the t axis, *i.e.* by the slope of the curve at that point. Thus, if when the change in concen-

tration is x_1 the angle made by the tangent with the t axis is ϕ_1, and when the change in concentration is x_2 the angle made by the tangent is ϕ_2, then the velocities at these two points are given by $\tan\phi_1$ and $\tan\phi_2$ respectively $\left(\text{since } \tan\phi = \frac{dx}{dt}\right)$, and can be read off directly from the graph.

$$\begin{aligned} \text{Thus} \quad & \tan\phi_1 = \mathrm{K}(a - x_1)^n \\ \text{and} \quad & \tan\phi_2 = \mathrm{K}(a - x_2)^n \end{aligned} \Bigg\} \text{ where } n = \text{order of reaction.}$$

$$\therefore \quad \frac{\tan\phi_1}{\tan\phi_2} = \frac{(a - x_1)^n}{(a - x_2)^n}$$

$$\therefore \quad \log\frac{\tan\phi_1}{\tan\phi_2} = n\log\frac{(a - x_1)}{(a - x_2)}$$

$$\text{whence} \quad n = \frac{\log\dfrac{\tan\phi_1}{\tan\phi_2}}{\log\dfrac{(a - x_1)}{(a - x_2)}} = \frac{\log\dfrac{u_1}{u_2}}{\log\dfrac{(a - x_1)}{(a - x_2)}}$$

(where $u_1 = \tan\phi_1$ and $u_2 = \tan\phi_2$).

As n must be an integer, it must be taken as the integer nearest to the value given by the right-hand side of the equation.

The two great objections to this method are:

(i) Errors may occur in drawing the curve.

(ii) There is great difficulty in reading off the value of the slope at any point with sufficient accuracy.

(4) Still another method is to "start the reaction with equivalent quantities of the reacting substances and determine in two experiments (which differ in concentration) the time required to consume half of the substances" (Nernst).

In the case of a *unimolecular reaction* the *time is independent* of the original concentration (Chapter XV., p. 260).

In the case of a *multimolecular reaction* the *time is inversely proportional to* a^{n-1}, where n is the order of the reaction (Chapter XV., p. 264).

$$\text{Thus,} \qquad \text{if } n = 2, \quad t \propto \frac{1}{a};$$

$$\text{if } n = 3, \quad t \propto \frac{1}{a^2};$$

$$\text{if } n = 4, \quad t \propto \frac{1}{a^3};$$

etc.

EXAMPLES.

(1) Madsen and Famulener in an investigation of the attenuation of vibriolysin at 28° C. found the following results:—

Time in Minutes.	Concentration.
0	100
10	78·3
20	67·6
30	59·3
40	49·8
50	40·8
60	34·4

Find the **order** of this reaction.

If we take the logarithms of the concentrations and plot them against time, we obtain the following plotting table:—

t . .	0	10	20	30	40	50	60
$\log_{10} C$.	2	1·894	1·83	1·773	1·697	1·61	1·537

The graph is a straight line (fig. 142), hence the reaction is unimolecular.

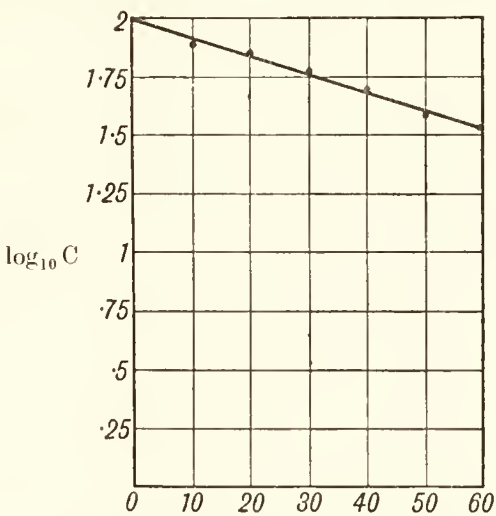


Fig. 142.—Attenuation of Vibriolysin.

(2) Miss Chick and Professor Martin studied the coagulation of hæmoglobin at various temperatures and obtained the following results at 70·4° C.:—

t (in mins.) . .	0	2	4	6	7·5
C (concentration of Hb)	100	52·5	25·3	14·1	7·6

To which order does this reaction belong?
What is the value of K?

By taking the logarithms of C we get the following table:—

t . .	0	2	4	6	7·5
$\log_{10}$ C .	2	1·72	1·40	1·15	0·88

When log C is plotted against t, the resulting graph is a straight line (fig. 143).

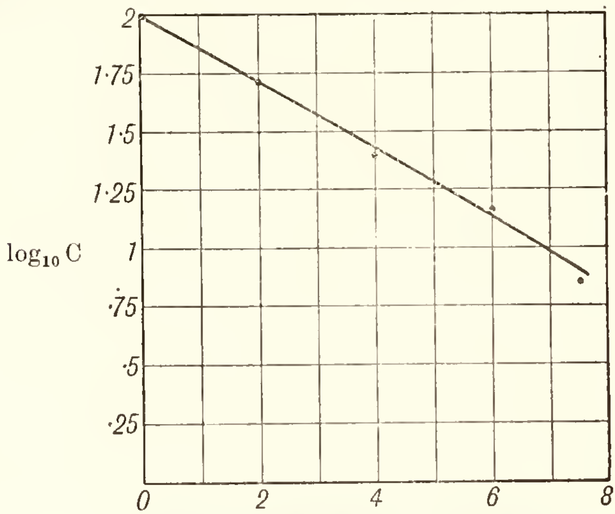


FIG. 143.—Coagulation of Hæmoglobin.

Therefore the coagulation proceeds as a monomolecular reaction.
Now the equation of a monomolecular reaction is

$$K = \frac{1}{t} \log \frac{a}{a-x} = \frac{1}{t}[\log a - \log (a-x)],$$

where a = initial concentration = C_0 (= 100 per cent.),
and $(a-x)$ = concentration at any time t = C_t.

$$\therefore \quad K = \frac{1}{t}(2 - \log C_t).$$

By taking any pair of corresponding values of t and log C_t we can therefore find K.

Thus, when $t = 2, \log C_t = 1{\cdot}72.$

$\therefore \quad K = \frac{1}{2}(2 - 1{\cdot}72) = 0{\cdot}140.$

When $t = 4, \log C_t = 1{\cdot}40.$

$\therefore \quad K = \frac{1}{4}(2 - 1{\cdot}40) = 0{\cdot}150.$

When $t = 6, \log C_t = 1{\cdot}15.$

$\therefore \quad K = \frac{1}{6}(2 - 1{\cdot}15) = 0{\cdot}142.$

When $t = 7{\cdot}5, \log C_t = 0{\cdot}88.$

$$\therefore \quad K = \frac{1}{7{\cdot}5}(2 - 0{\cdot}88) = 0{\cdot}149.$$

Hence K is practically constant (within the limits of experimental error), again proving that the reaction is unimolecular.

(3) Victor Henri found the following figures in the case of hæmolysis of chicken erythrocytes by means of normal serum. Prove that the reaction proceeds in accordance with the equation for a unimolecular reaction.

Quantity of serum = 0·3 c.c.					
	t (in mins.) .	24	63	94	190
	x (hæmolysis)	33%	56%	78%	96%

Using the equation for a unimolecular reaction, viz. $K = \frac{1}{t} \log \frac{a}{a - x}$ (where a = original concentration = 100 per cent.), and substituting various corresponding values of t and x we find that K is the same in each case, as follows:—

	$\frac{1}{t} \log \frac{100}{100 - x} = K.$	
$t = 24$, $x = 33$	$\frac{1}{24} \log \frac{100}{67}$	$\frac{1}{24}(2 - 1{\cdot}826) = 0{\cdot}0072$
$t = 63$, $x = 56$	$\frac{1}{63} \log \frac{100}{44}$	$\frac{1}{63}(2 - 1{\cdot}643) = 0{\cdot}0057$
$t = 94$, $x = 78$	$\frac{1}{94} \log \frac{100}{22}$	$\frac{1}{94}(2 - 1{\cdot}342) = 0{\cdot}0070$
$t = 190$, $x = 96$	$\frac{1}{190} \log \frac{100}{4}$	$\frac{1}{190}(2 - 0{\cdot}602) = 0{\cdot}0073$

Therefore the reaction proceeds in accordance with the equation for a unimolecular reaction.

We could have shown the same thing graphically by plotting t against $\log (a - x)$ thus:

t	24	63	94	190
$\log (a - x)$	1·826	1·644	1·342	0·602

The result is a straight line.

(4) Madsen and Walbum studied the progress of tryptic digestion by

subjecting 10 grm. of casein powder to the action of 100 c.c. of a 1 per cent. solution of trypsin at constant temperature and testing the amount of remaining casein $(a-x)$ at various times t by means of nitrogen determinations (by Kjeldahl's method). The following figures represent the results found.

Find whether these figures agree with the supposition that the process is a bimolecular reaction.

Time (hours) .	0	0·5	2·5	6	11	24	33	48	72
Nitrogen concentration	0·11	0·108	0·102	0·1	0·096	0·076	0·07	0·06	0·049

If the process is a bimolecular reaction, then t, when plotted against the reciprocal of the nitrogen concentration, should give a straight line. The plotting table would be:

t .	0	0·5	2·5	6	11	24	33	48	72
$\frac{1}{N}$.	9	9·3	9·8	10·0	10·4	13·2	14·3	16·7	20·4

The graph (see fig. 144) is practically a straight line, therefore the process goes on as a bimolecular reaction.

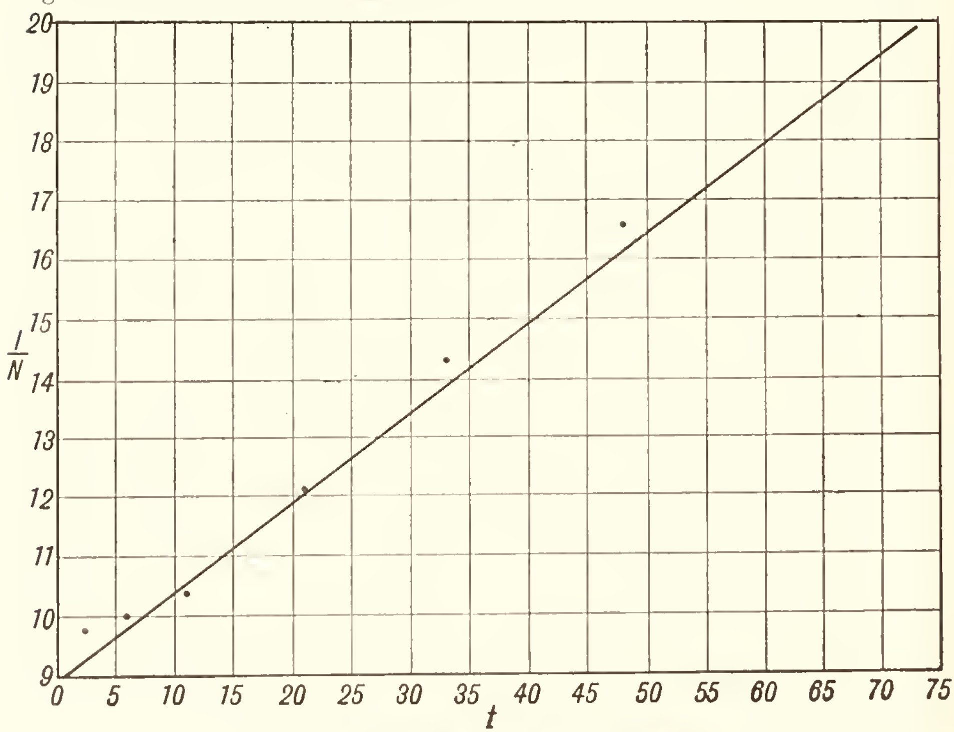


FIG. 144.—Results on Tryptic Digestion.

(5) Madsen and Walbum studied the decomposition of tetanolysin by means of peptone. Quantitative examinations would suggest the following scheme under which the reaction occurs, viz.:—

Two molecules of tetanolysin + 3 molecules of peptone
= lysinpeptone.

Investigate if this scheme is correct, the following results being given in the case of a certain experiment:—

Time in Hours.	Concentration of Tetanolysin.
0·5	47·7
1	39·7
2	30·3
4	22·3
6	18·1
8	17·0

In this case by plotting log C or $\frac{1}{C^2}$, $\frac{1}{C^3}$, $\frac{1}{C^4}$, etc., against t the resulting graphs are not straight lines, but by plotting $\frac{1}{C}$ against time according to the following table there results practically a straight line (fig. 145):—

t .	0·5	1	2	4	6	8
$\frac{1}{C}$.	0·021	0·025	0·033	0·045	0·055	0·059

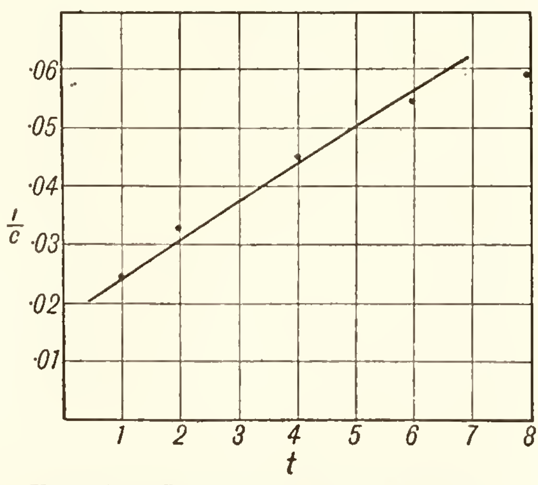


FIG. 145.—Decomposition of Tetanolysin.

Hence the reaction is bimolecular, not pentamolecular as the scheme suggests. The reason is that the lysinpeptone formed is very unstable and

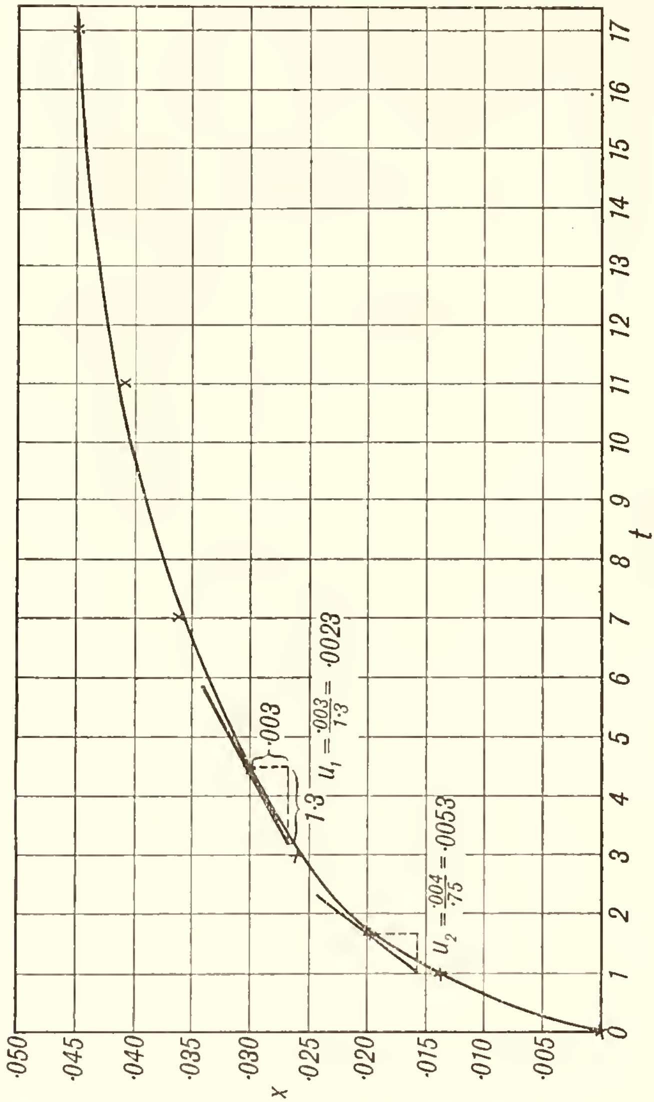


Fig. 146.

decomposes as soon as it is formed with reformation of peptone, so that the concentration of peptone remains constant. Hence in the equation:

$$\text{Reaction velocity} = K(C_T)^2 \, . \, (C_P)^3.$$

But C_P being constant, we get

$$\text{Reaction velocity} = K'(C_T)^2,$$

the equation for a bimolecular reaction.

(6) The following values of x and $(c-x)$ were observed at the given times t. Find the order of the reaction by method (3).

t.	x.	$c-x$.
1	0·01434	0·04816
1·75	0·01998	0·04252
3	0·02586	0·03664
4·5	0·03076	0·03174
7	0·03612	0·02638
11	0·04102	0·02148
17	0·04502	0·01748

The graph (t against x) is shown in fig. 146. u_1 is seen to be $= 0·0023$ and $u_2 = 0·0053$, where u_1 and u_2 are the slopes of the curve at two points (see p. 372).

$$\therefore \qquad \frac{u_1}{u_2} = \frac{23}{53} = 0·434.$$

$$\therefore \quad \log\frac{u_1}{u_2} = \log 0·434 = \bar{1}·6375 = -0·3625.$$

$$(c-x_2) = 0·04252, \quad \text{and} \quad (c-x_1) = 0·03174.$$

$$\therefore \qquad \frac{c-x_1}{c-x_2} = \frac{3174}{4252} = 0·746,$$

$$\therefore \quad \log\left(\frac{c-x_1}{c-x_2}\right) = \log 0·746 = \bar{1}·8727 = -0·1273,$$

$$\therefore \qquad n = \frac{3625}{1273} = 2·8 = \mathbf{3} \text{ (to nearest integer)},$$

$\therefore$ Reaction is trimolecular.

(7) The following values of x and $(c-x)$ were found in the case of the destruction of vibriolysin by peptone. Find the order of the reaction.

t.	x.	$c-x$.
0·5	52·3	47·7
1	60·3	39·7
2	69·7	30·3
4	77·7	22·3
6	81·9	18·1
8	83	17

The graph is shown in fig. 147.

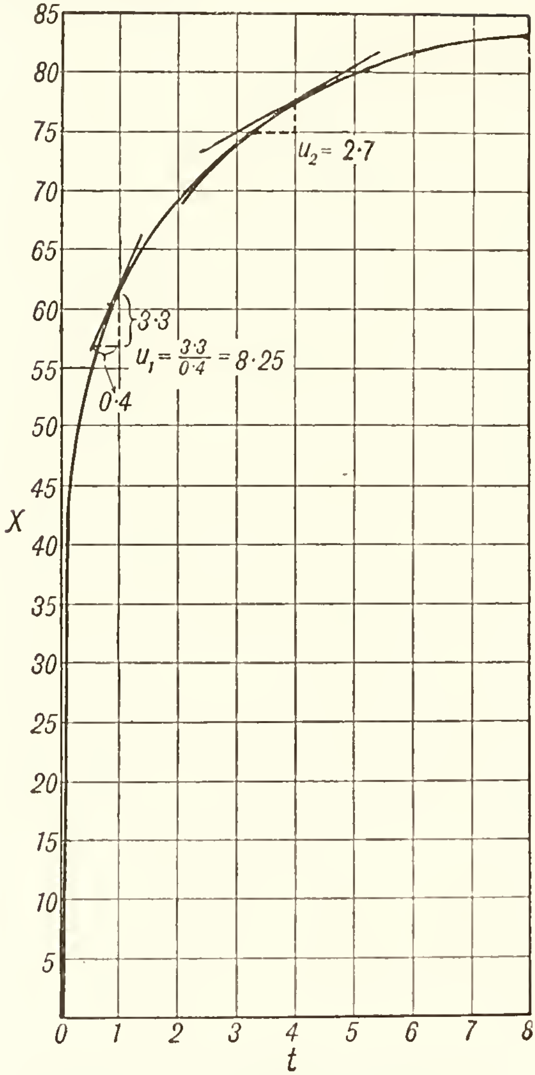


FIG. 147.—Decomposition of Vibriolysin.

$$u_1 = 8{\cdot}25, \quad \text{and} \quad u_2 = 2{\cdot}7,$$

$$\therefore \quad \frac{u_1}{u_2} = \frac{8{\cdot}25}{2{\cdot}7} = 3{\cdot}06;$$

$$\log 3{\cdot}06 = 0{\cdot}4857.$$

$$(c - x_1) = 39{\cdot}7, \quad (c - x_2) = 22{\cdot}3,$$

$$\therefore \quad \frac{c - x_1}{c - x_2} = \frac{39{\cdot}7}{22{\cdot}3} = 1{\cdot}8,$$

$$\therefore \quad \log \frac{(c - x_1)}{(c - x_2)} = \log 1{\cdot}8 = 0{\cdot}2553;$$

$$\therefore \qquad n = \frac{4857}{2553} = 1{\cdot}9.$$

$\therefore$ Reaction is bimolecular.

Empirical Formulæ Connecting more than Two Variables.—Sometimes it is necessary in biological work to find an empirical formula connecting more than two variables. The process then becomes considerably more complicated and difficult. We shall take as an example the method by which D. and E. F. Du Bois found their formula connecting the surface area of the body with the height and weight of a person,

viz., $S = 71{\cdot}8W^{0{\cdot}425} \,.\, H^{0{\cdot}725}$,

where S = surface in square centimetres,

W = weight in kilogrammes,

H = height in centimetres.

The surface area being bidimensional, it may be put $= L^2$. Weight being tridimensional, let it be put $= L^3$, and height being unidimensional, it may be put $= L$. It is clear, therefore, that any formula connecting these variables must be of the form

$$S = K \,.\, W^{\frac{1}{m}} \,.\, H^{\frac{1}{n}},$$

where K is a constant, and m and n are such that $\frac{3}{m} + \frac{1}{n} = 2$.

For by taking logarithms of both sides we get

$$\log S = \log K + \frac{1}{m} \log W + \frac{1}{n} \log H,$$

or

$$\log L^2 = K + \frac{1}{m} \log L^3 + \frac{1}{n} \log L$$

(where K stands for the constant log K),

i.e.

$$2 \log L = \frac{3}{m} \log L + \frac{1}{n} \log L + K,$$

$$\therefore \quad 2 = \frac{3}{m} + \frac{1}{n}$$ (the K is ignored as it is not a dimension).

Possible values of $\frac{1}{m}$ and $\frac{1}{n}$ are $\frac{1}{3}$ and 1, or $\frac{1}{2}$ and $\frac{1}{2}$

$$\left(\text{thus } \frac{3}{3} + 1 = 2, \quad \text{and} \quad \frac{3}{2} + \frac{1}{2} = 2\right).$$

Let us take $\frac{1}{m} = \frac{1}{3}$ and $\frac{1}{n} = 1$.

The formula then becomes

$$S = K\,W^{\frac{1}{3}} \,.\, H \quad \text{or} \quad K = \frac{S}{W^{\frac{1}{3}}H}$$

The following are a few of the values found by actual measurement:—

	(1)	(2)	(3)	(4)	(5)
S	8473	16720	12320	20760	14907
W	24·2	64·0	36·5	87·1	45·2
H	110·3	164·3	146·0	182·8	171·8

From this table we get (by using formula $S = KW^{\frac{1}{3}}H$ or $K = \frac{S}{W^{\frac{1}{3}}H}$):

$$\left.\begin{array}{l}(1)\ K = 26{\cdot}6\\ (2)\ K = 25{\cdot}5\\ (3)\ K = 25{\cdot}4\\ (4)\ K = 25{\cdot}6\\ (5)\ K = 24{\cdot}4\end{array}\right\}\text{Average is } K = 25{\cdot}5.$$

By using the formula $S = KW^{\frac{1}{2}}H^{\frac{1}{2}}$,

$$i.e.\ K = \frac{S}{W^{\frac{1}{2}}H^{\frac{1}{2}}}$$

the values of K found by using the above table of values of S, W and H were:

$$\left.\begin{array}{l}(1)\ K = 164{\cdot}0\\ (2)\ K = 163{\cdot}0\\ (3)\ K = 168{\cdot}9\\ (4)\ K = 164{\cdot}5\\ (5)\ K = 169{\cdot}2\end{array}\right\}\text{Average is } 165{\cdot}9.$$

Now it is seen that in case (5), for instance, the percentage variation of K from the corrected mean value is positive when the first formula is used, but negative when the second formula is used. Hence, the indication is that $\frac{1}{m}$ must be greater than $\frac{1}{3}$ and less than $\frac{1}{2}$, whilst $\frac{1}{n}$ must be less than 1 and greater than $\frac{1}{2}$ (always providing $\frac{3}{m}+\frac{1}{n} = 2$).

Supposing one takes $\quad \frac{1}{m} = \frac{1}{2{\cdot}5} = \frac{2}{5}$.

This would make $\quad \frac{1}{n} = 2 - \frac{3 \times 2}{5} = \frac{4}{5}$.

The resulting value of K would then be $\frac{S}{W^{0{\cdot}4}H^{0{\cdot}8}}$.

By using this formula better results are obtained than by means of either of the other two previous formulæ.

The best value is found to be when

$$\frac{1}{m} = \frac{1}{2{\cdot}35} = 0{\cdot}425$$

and
$$\frac{1}{n} = \frac{1}{1{\cdot}38} = 0{\cdot}725$$

$$\left(\frac{3}{m} + \frac{1}{n} = 3 \times 0{\cdot}425 + 0{\cdot}725 = 1{\cdot}275 + 0{\cdot}725 = 2\right),$$

∴ Formula is

$$S = KW^{0{\cdot}425}H^{0{\cdot}725}$$

and
$$K = \frac{S}{W^{0{\cdot}425}H^{0{\cdot}725}}$$

$$\left.\begin{array}{l}(1)\ K = 71{\cdot}30 \\ (2)\ K = 70{\cdot}65 \\ (3)\ K = 72{\cdot}01 \\ (4)\ K = 71{\cdot}22 \\ (5)\ K = 70{\cdot}36\end{array}\right\}\text{ The corrected average was found to be } 71{\cdot}84.$$

∴ Formula is $S = 71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}$.

(See p. 146 *et seq.*, for a nomogram for this equation.)

EXERCISES.

(1) Find the law connecting x and y from the following observations (allowing for errors of observation):—

x	2·5	3·5	4·4	5·8	7·5	9·6	12·0
y	13·5	17·6	22·2	28·0	35·5	47·4	56·1

[*Answer*, $y = 1{\cdot}42 + 4{\cdot}66x$.]

(2) The following values have been found for x and y:—

x	4	5	6	7	8	9	10	11
y	6·29	5·72	5·22	4·78	4·35	4·06	3·75	3·48

It is found that the following two empirical formulæ seem to be nearly equally good:—

$$y = \frac{a}{b+x}, \quad \text{and} \quad y = ae^{-\beta x}.$$

Find the best values of a and b, a and β.

[*Answer*, $a = 54{\cdot}53$, $b = 4{\cdot}67$; $a = 8{\cdot}706$, $\beta = 0{\cdot}084$.]

(3) The following values of x and y are believed to be related by an equation of the form $y = Ae^{bx}$. Examine if this is so and calculate A and b.

x	0·1	0·2	0·3	0·4	0·5	0·6
y	0·4254	0·4093	0·3704	0·3352	0·3032	0·2744

[*Answer*, $y = 0{\cdot}5e^{-x}$.]

(4) Prove that the following values of x and t, found in the case of plant protease, are in agreement with the Schütz-Borissoff law (x = milligrammes, t = hours):—

x	5·34	8·42	9·82	11·92	12·98	13·70	17·22
t	1	2	3	4	5	6	9

[*Answer*, $\dfrac{x}{\sqrt{t}}$ is practically constant (5·34 to 5·74).]

(5) The following numbers represent the relation between the weight in kilogrammes (W) and the surface in square decimetres (S) of a number of children. If the law connecting S and W is $S = AW^m$, find the values of A and m. How much milk would an infant weighing 2·50 kilogrammes require per day if the amount of heat lost by the body is 1700 calories per square metre per day, and the calorific value of milk is 736 calories per litre?

W	2	3	4	5	6	7	8	9	10
S	16·3	21·4	26·0	30·1	34·0	37·7	41·2	44·6	47·9

[*Answer*. Plot log W against log S; the resulting graph is the straight line $\log S = \log A + m \log W$. m will be seen at once to be 2/3, and A can be found by substitution to be 10·3,

$$\therefore \quad \text{Law is } S = 10{\cdot}3\sqrt[3]{W^2}.$$

When W = 2·50, S becomes 19 (either by calculation or by interpolation), *i.e.* = 0·19 square metre,

$$\therefore \quad \text{Amount of milk required is } \frac{0{\cdot}19 \times 1700}{736} \text{ litres} = 440 \text{ c.c.}]$$

(6) The following are the pulse-rates P of people of different height H:—

H (in cms.)	50	69·8	79·6	86·7	98·6	167·5 (= height of adult)
P . .	134	111	108	104	98	73 (rate in adult)

Find whether an equation of the form $P = AH^m$ (where A and m are constants) will express the relationship between P and H, and evaluate the constants.

[Plotting log P against log H there results a straight line for which $m = -\frac{1}{2}$.

$\therefore \quad P = \dfrac{A}{\sqrt{H}}$. When P = 73, H = 167·5; $\therefore \quad 73 = \dfrac{A}{\sqrt{167 \cdot 5}}$;

$\therefore \quad A = 73\sqrt{167 \cdot 5}$; $\quad \therefore$ Equation is $P = 73\sqrt{\dfrac{167 \cdot 5}{H}}$.]

(7) The following have been found to be the rates of the rabbit's heart (r) at the given temperatures (t). Allowing for errors of observation, find by plotting whether these figures are in agreement with Van't Hoff-Arrhenius law. [Assume T_1T_2 to be constant (see p. 281).]

t° C.	0·4	5·6	6·4	12·8	13·6	14·0	16·0
r .	5·9	11·7	12·3	25·9	28·4	29·7	37·8

[*Answer.* When T_1T_2 is constant, the Van't Hoff-Arrhenius law in this case becomes

$$r_2/r_1 = 10^{A(T_2-T_1)} \text{ (see p. 281)}$$

or $$\log r_2 - \log r_1 = A(T_2 - T_1).$$

Hence if the above figures obey the law, then $\log\left(\dfrac{r_2}{r_1}\right)$ plotted against $(T_2 - T_1)$ should give a straight line. This will be found to be the case.]

(W. M. Feldman and A. J. Clark, *The Lancet*, 1924, vol. i.)

(8) Show from the figures in Exercise (6) on p. 95, that disinfection takes the form of a unimolecular reaction.

(9) Miss Chick tested the bactericidal value of 5 per cent. phenol at 33·3° C. by mixing a number of anthrax spores with a certain amount of the disinfectant and estimating the number of bacteria in one drop of the mixture at various intervals of time. She obtained the following results:—

Number of bacteria (n) .	439	275·50	137·50	46	15·8	5·45	3·5	0·5
Time in hours (t) . .	0	0·50	1·25	2	3	4·1	5	7

Draw the appropriate graph and state what inference can be drawn therefrom. How would you ascertain whether the graph obeys the logarithmic law, and the order of the chemical reaction of the bactericidal process? Calculate the value of the reaction constant.

[*Answer.* The curve is a smooth and regular one, showing that the relation between the variables can be represented by some simple mathematical function (p. 107). Plotting t against $\log n$, the points lie almost exactly in a straight line, so that the relationship between n and t is a logarithmic one of the form $n_t = n_0 e^{-Kt}$ or $K = \frac{1}{t} \log \frac{n_0}{n_t}$. Hence disinfection proceeds as a unimolecular reaction.

For $t = 0{\cdot}5$ hr., $K = 2(\log 439 - \log 275{\cdot}5) = 2(2{\cdot}64 - 2{\cdot}44) = 0{\cdot}40$.

„ $t = 1{\cdot}25$ hrs., $K = \frac{1}{1{\cdot}25}(\log 439 - \log 137{\cdot}5) = \frac{2{\cdot}64 - 2{\cdot}14}{1{\cdot}25} = 0{\cdot}40$,

and so on, giving K a mean value of 0·44.]

(10) Scammon and Calkins found the following relationship to hold good between the length of a fœtus in cms. (l) and its age in lunar months (T), viz. $T = \left(\frac{l}{28} + 1{\cdot}25\right)^2 + 0{\cdot}74$. Estimate the age of a fœtus 30·2 cms. long.

[*Answer.* 6·17 months.]

(11) Starting with 1760 bacilli, it was found that they reached a limit of 106,000,000 in 8 hours. Assuming ab (see Example (6), p. 335 and p. 363) to have been found to be 1·87, find the theoretical number to which the bacteria should have grown in 4 hours.

[*Answer.* $$y = \frac{106{,}000{,}000}{\frac{106{,}000{,}000}{1760} e^{-7{\cdot}48} + 1}$$
$$= 3{,}030{,}000.]$$

CHAPTER XXIII.

BIOMETRY.

As explained in the opening chapter of this book none of the quantitative results obtained in the laboratory can ever be absolutely exact. Even in the domain of the so-called exact sciences, such as astronomy, physics and chemistry, where methods of measurement are as near perfection as human ingenuity can achieve, and where the object measured is of constant size, it is found that several apparently equally reliable measurements taken of the same quantity even by the same observer are never exactly the same. Thus, notwithstanding the exceedingly skilful and minute precautions taken by Crookes to ensure accuracy in his atomic weight determinations of Thallium, in a series of twelve results the values ranged between 203·628 and 203·666, with an average of 203·642. Indeed, too close an agreement between a number of measurements is open to the suspicion of being too good to be true. The discrepancies are due to **errors of observation** inherent in the observer. When we come to deal with measurements of living things, we are, in addition to the errors of observation, faced with the difficulty that the thing we are measuring is not constant, but variable. No two human beings, for instance, of exactly the same age are ever of exactly the same height or weight. In other words, the biological investigator, in addition to the actual errors of observation, is confronted with the discrepancies resulting from the inherent variability of the objects he measures. One of the objects of modern statistical method is to discover what is called "**the most probable result**" out of a series of discordant results obtained in the laboratory.

Biometry is the application of modern statistical methods to the measurements taken of biological (variable) objects.

[The reader must realise that the term "discordant" is not synonymous with "faulty." Provided the measurements have been made with every possible care, and with accurate apparatus, the divergences between the results are merely the effect of errors of observation over which the observer has no control, and the application of statistical methods to such discordant

results will tend to minimise the effect of these errors on the final result. But no amount of statistical mathematics—however ingenious and exact—will elicit correct conclusions from sets of measurements that have been made by careless observers or with faulty apparatus or methods. Hence when a laboratory worker collects his results for statistical treatment he must not abstain from including any result however much it diverges from the other results if that result has been honestly and carefully obtained. If, however, a bad result has been obtained on account of careless arithmetic (such as faulty addition or multiplication, etc.), or faulty measuring apparatus (such as an incorrectly calibrated pipette, etc.) it must be ignored.]

Principles of Probability.—As the whole modern theory of statistics is based upon the theory of probability, it is necessary to discuss and elucidate a few of the more elementary points in connection with the subject of *probability* before we can consider the subject of biometry, with which we are concerned in this chapter.

Definition.—Events which occur as the result of circumstances over which we have no control are called "random" or "chance" events, and the degree of likclihood with which we can predict the occurrence of any one such event is taken as the measure of the probability of such an event happening.

If, for instance, an event may happen in a ways and fail in b ways, then the probability or chance of its happening is $\frac{a}{a+b}$, and that of its failing is $\frac{b}{a+b}$.

Suppose a bag contains a dozen perfectly round balls made of the same material, size and weight, and of the same degree of smoothness, but that 6 of the balls are white and 6 are black. It is clear that the probability of a person drawing a white ball (without looking) is 6/12, and the probability of his drawing a black ball (or of his failing to draw a white one) is also 6/12. Similarly, if 7 of the balls are white and 5 are black, the probability of drawing a white ball would be 7/12 and that of failing to draw a white one 5/12. We do not, of course, mean that out of 12 draws the proportion of white and black balls drawn would be 6 : 6 (or 1 : 1) in the first case, and 7 : 5 in the second case. What we do mean is that out of a very large number, say, 12,000 draws, the proportion of white to black balls would be very nearly 1 : 1 in the first case and 7 : 5 in the second case, and the greater the number of draws, the more nearly will whites and blacks approach these proportions. In general terms, therefore, we may say that if an event may happen in a ways (*e.g.* 7) and fail in b ways (*e.g.* 5), then the probability of its happening is $\frac{a}{a+b}$ $\left(e.g.\ \frac{7}{7+5} = \frac{7}{12}\right)$,

and that of its failing is $\frac{b}{a+b}$ $\left(e.g.\ \frac{5}{7+5} = \frac{5}{12}\right)$.

Note.—Since an event must either happen or fail to happen, *e.g.* a draw in the foregoing case will be certain to bring forth either a white or a black ball,

$$\therefore \quad Certainty = \frac{a}{a+b} + \frac{b}{a+b} = \frac{a+b}{a+b} = 1.$$

Hence if we indicate the probability of an event happening by p, and that of its failing by q, then it follows that

$$p+q = 1 = \text{certainty},$$

and

$$q = 1-p.$$

As we fix unity as the expression for eertainty, we express *no possibility* by zero. Between 0 and 1 we have an infinity of fraetions whieh express an infinite number of ehanees.

Thus, out of a bag eontaining white balls only, one is eertain to draw a white ball, so that the probability of sueh an event happening is 1. On the other hand, the probability of drawing a blaek ball is zero. If the bag eontains both white and blaek balls, the probability of drawing a ball of a partieular eolour depends upon the relative proportions of the two kinds of balls.

LAW I.—**The probability of the occurrence of any one of several exclusive events (*i.e.* of events which cannot occur together) is equal to the sum of the separate probabilities.**

Suppose a bag eontains 25 balls, 3 of whieh are white, 4 blaek and 18 red, what is the probability that the first ball drawn will be a white or a blaek one?

The probability of the first ball being white is 3/25; that of the first ball being blaek is 4/25; therefore the probability of the first ball being either white or blaek is $(3+4)/25 = 7/25$. Similarly, the probability of the first ball being either white or red is $(3+18)/25 = 21/25$.

LAW II.—**The probability of the occurrence together of two independent events is the product of their separate probabilities.**

Suppose we have two separate bags, one of which contains, say, a dozen balls, of whieh 5 are white and 7 black, and the other eontains 25 balls, out of whieh 3 are white and 22 are black. What is the probability that in two draws, one from eaeh of the two bags, a white ball will appear from each?

It is obvious that the probability in this case of the two events happening together must be smaller than the probability of

the happening of one event alone. In fact, the probability of a white ball coming out at the first draw in each case must be the product of the two separate probabilities, viz. $5/12 \times 3/25 = 1/20$.

(i) To show that this must be so, we will take the simple case of two bags each containing one white (W) and one black (B) ball. When a draw is made from each bag at random the following events are possible:—

(1) W from 1st bag and W from 2nd bag; (2) W from 1st bag and B from 2nd bag; (3) B from 1st bag and W from 2nd bag; (4) B from 1st bag and B from 2nd bag. Hence out of four possible results, the event of drawing two whites from the two bags can occur once only, *i.e.* the probability of such an event occurring is 1/4. As the probability of drawing W from each bag separately is 1/2 in each case, it is seen that the probability of the two independent events happening together is the product of the separate probabilities, viz. $1/2 \times 1/2$.

(ii) Similarly, if each of the bags contains, say, one white and two black balls, called respectively W, w, B_1, B_2, b_1, b_2 (the capital letters representing the balls in one bag and the small letters those in the other), the possible events are [(Ww), (Wb_1), (Wb_2)]; [(B_1w), (B_1b_1), (B_1b_2)]; [(B_2w), (B_2b_1), (B_2b_2)]; so that the probability of drawing a white ball from each bag (Ww) is 1/9, *i.e.* the product of the separate probabilities $1/3 \times 1/3$. The probability of drawing two blacks is $2/3 \times 2/3 = 4/9$.

(iii) If one bag contains one white and one black (W and B), and the other contains one white and two blacks (w, b_1 and b_2), the possible events are [(Ww), (Wb_1), (Wb_2)]; [(Bw), (Bb_1), (Bb_2)]; so that the probability of drawing a white ball from each bag (Ww) is 1/6, which is again the product of the separate probabilities, viz. $1/2 \times 1/3$; and so on.

If in (ii) the two drawings, instead of being made from two separate bags each containing one white and two black balls, were made from one bag only, *the ball drawn at the first trial being replaced before the second trial*, the two possible events would still be independent and the probability of drawing two successive whites would still be $1/3 \times 1/3 = 1/9$, while that of drawing two successive blacks would similarly still be $2/3 \times 2/3 = 4/9$. If, however, *the ball drawn at the first trial is not replaced*, then the probability of drawing a white or black ball at the second trial obviously depends upon the colour of the ball drawn at the first trial. The second event then becomes dependent upon the first event, and the probability of the two events occurring successively is no longer $1/3 \times 1/3$ in

the case of whites and $2/3 \times 2/3$ in the case of blacks. Thus if a white ball was drawn at the first trial and not replaced, no white can be drawn at all at the second trial since there are no whites left in the bag, so that the probability of drawing two whites is zero, while if a black ball be drawn at the first trial and not replaced the probability of drawing another black at the second trial is $1/2$ instead of $2/3$, so that the probability of drawing two black balls is now $2/3 \times 1/2 = 1/3$, instead of $2/3 \times 2/3 = 4/9$. We see, therefore, that the probability of the occurrence together of two dependent events is different from that in the case of two independent events.

EXAMPLES.

(1) Out of 32,573 patients on a hospital register, 1979 had biliary disease and 543 had glycosuria, while 33 had both these diseases. There were 1237 Jewish patients on the register, of whom 56 had glycosuria. What is the relationship between (i) biliary disease and glycosuria; (ii) race and glycosuria?

Probability of a patient having biliary disease $= \dfrac{1{,}979}{32{,}573} = 0{\cdot}0608.$

" " " " glycosuria $= \dfrac{543}{32{,}573} = 0{\cdot}0167.$

" " " being a Jew $= \dfrac{1{,}237}{32{,}573} = 0{\cdot}0379.$

$\therefore$ Probability of a patient with biliary disease having glycosuria is

$$0{\cdot}0608 \times 0{\cdot}0167 = 0{\cdot}0010,$$

and probability of a glycosuria subject being a Jew

$$= 0{\cdot}0379 \times 0{\cdot}0167 = 0{\cdot}0006.$$

Hence, by chance alone we would expect

(i) $32{,}573 \times 0{\cdot}0010 = 32$ patients to have both biliary disease and glycosuria, a number which is practically identical with the 33 actually found, showing the absence of causal relationship between the two diseases.

(ii) $32{,}573 \times 0{\cdot}0006 = 20$ Jewish patients to have glycosuria, instead of the 56 actually found, showing that Jews are particularly prone to glycosuria.

Note.—This example is purely hypothetical, intended to illustrate a principle, and is not an actual record of hospital statistics.

(2) The chance of a person A, aged 35, dying within 30 years is 9/16, and that of a person B, aged 45, dying within 30 years is 3/5. What is the chance that one at least of these persons will be alive 30 years hence?

Probability of both being dead 30 years hence $= \dfrac{9}{16} \times \dfrac{3}{5} = \dfrac{27}{80}.$

$\therefore$ Probability of both A and B not being dead, *i.e.* that at least one will be alive, is $1 - \dfrac{27}{80} = \dfrac{53}{80}.$

Possibility and Probability of Certain Events Occurring.—Let us take again the case of a bag containing balls of different colours. Suppose, for instance, it contains 10 balls, of which 5 are white and 5 are black, and that we make a number of successive draws (always returning the ball drawn before making another draw), what are the probabilities of the appearance of a certain number of whites and of another number of blacks in a given number of draws?

(1) In *one draw* there are 2 (*i.e.* 2^1) *possible events*, viz. white or black—each of which is equally probable. These possibilities may be denoted by

$$(W, B)$$

(where W and B are not numerical quantities, but only symbolical expressions denoting events).

(2) In *two draws* there are 4 (*i.e.* 2^2) *possible events*, which may be denoted by

$$(WW, WB, BW, BB)$$

which means that the W or B of the first draw may combine with the W or B of the second draw to give either white followed by white (WW), or white followed by black (WB), or black followed by white (BW), or black followed by black (BB). In other words, the different possibilities may be represented by the terms (together with their respective coefficients) of the binomial expansion $(W+B)^2 = W^2+2WB+B^2$.

(3) In *three successive draws* there are 8 (*i.e.* 2^3) *possible events*, because the 4 possible events of the first two draws may be combined with the 2 possible events of the third draw, thus

$$(W+B)(W^2+2WB+B^2) = W^3+3W^2B+3WB^2+B^3,$$

i.e. the 8 different possibilities will correspond to the terms (together with their respective coefficients) of the expansion $(W+B)^3$. It is to be noted that such terms as W^3, W^2B, etc., are short ways of expressing the fact that 3 white balls or 2 white and 1 black ball have been drawn, and that the numerical coefficient of each term denotes the number of different possible ways in which this event may happen.

(4) Similarly, in *four successive draws* there are 16 (*i.e.* 2^4) possibilities, corresponding to the terms of the expansion

$$(W+B)^4 = W^4+4W^3B+6W^2B^2+4WB^3+B^4.$$

(5) *In general*, in n *successive draws* there are 2^n possible events corresponding to the terms of the binomial expansion

$$(W + B)^n = W^n + nW^{n-1}B + \frac{n(n-1)}{2\,!}W^{n-2}B^2$$

$$+ \frac{n(n-1)(n-2)}{3\,!}W^{n-3}B^3 + \ldots + B^n.$$

[As an illustration of the working of the laws of probability in biology we might take **the case of Mendelian expectations** when both parents are impure dominants with respect to one unit character. We then have

$$(D + R)\,(D + R) = D^2 + 2DR + R^2,$$

so that the offspring will consist of pure dominants (D^2), impure dominants (DR) and recessives (R^2) in the proportion of 1 : 2 : 1. For the calculation of Mendelian expectations when the parents differ in more than one unit character the reader is referred to "*Child Physiology*," pp. 58–60.]

So much for the *possibilities*. Now what are the *probabilities* of the different events happening?

Let us take, for example, the case of 4 draws. There are 16 equally probable events, viz.

One of 4 successive whites (W^4).
Four of 3 whites and 1 black ($4W^3B$).
Six of 2 whites and 2 blacks ($6W^2B^2$).
Four of one white and 3 blacks ($4WB^3$).
One of 4 blacks (B^4).

Hence, the respective probabilities or frequencies of the above combinations are $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$ and $\frac{1}{16}$ respectively. In other words, the probability of any particular combination appearing (or its *frequency*) corresponds to the respective term of the binomial expansion $(\frac{1}{2} + \frac{1}{2})^4$.

Similarly, the frequency for each of the combinations in 5 draws would correspond to the respective term of the expansion $(\frac{1}{2} + \frac{1}{2})^5$, and, in general, the frequency of each combination

happening in n draws would be represented by the eorresponding term of the binomial expansion $(\frac{1}{2}+\frac{1}{2})^n$, viz.

$$\left(\frac{1}{2}\right)^n, \frac{n}{2^n}, \frac{n(n-1)}{2^n \,.\, 2\,!}, \frac{n(n-1)\,(n-2)}{2^n \,.\, 3\,!}, \ldots \frac{n}{2^n}, \frac{1}{2^n}.$$

Thus, if $n = 10$, *i.e.* if we make 10 draws from a bag eontaining equal numbers of white and blaek balls (replaeing the ball after eaeh draw), the frequeneies of the possible results will be the sueeessive terms of the expansion $(\frac{1}{2}+\frac{1}{2})^{10}$, as follows:—

All balls white,	$\frac{1}{2^{10}}$	$= \frac{1}{1024}$	$= 0{\cdot}000977$,
9 whites, 1 blaek,	$\frac{10}{2^{10}}$	$= \frac{10}{1024}$	$= 0{\cdot}009766$,
8 whites, 2 blaeks,	$\frac{10\,.\,9}{2^{10}\,.\,2\,!}$	$= \frac{45}{1024}$	$= 0{\cdot}043945$,
7 whites, 3 blaeks,	$\frac{10\,.\,9\,.\,8}{2^{10}\,.\,3\,!}$	$= \frac{120}{1024}$	$= 0{\cdot}117187$,
6 whites, 4 blaeks,	$\frac{10\,.\,9\,.\,8\,.\,7}{2^{10}\,.\,4\,!}$	$= \frac{210}{1024}$	$= 0{\cdot}205078$,
5 whites, 5 blaeks,	$\frac{10\,.\,9\,.\,8\,.\,7\,.\,6}{2^{10}\,.\,5\,!}$	$= \frac{252}{1024}$	$= 0{\cdot}246094$,

whieh is the middle term of the expansion.

The frequeneies then deerease in regular order (see p. 67),

4 whites, 6 blaeks,	$= 0{\cdot}205078$,
3 whites, 7 blacks,	$= 0{\cdot}117187$,
2 whites, 8 blaeks,	$= 0{\cdot}043945$,
1 white, 9 blacks,	$= 0{\cdot}009766$,
All balls black,	$= 0{\cdot}000977$.
	$1{\cdot}000000$

These frequencies, or probabilities, are plotted as ordinates at equal intervals along a horizontal axis as abscissa in fig. 148.

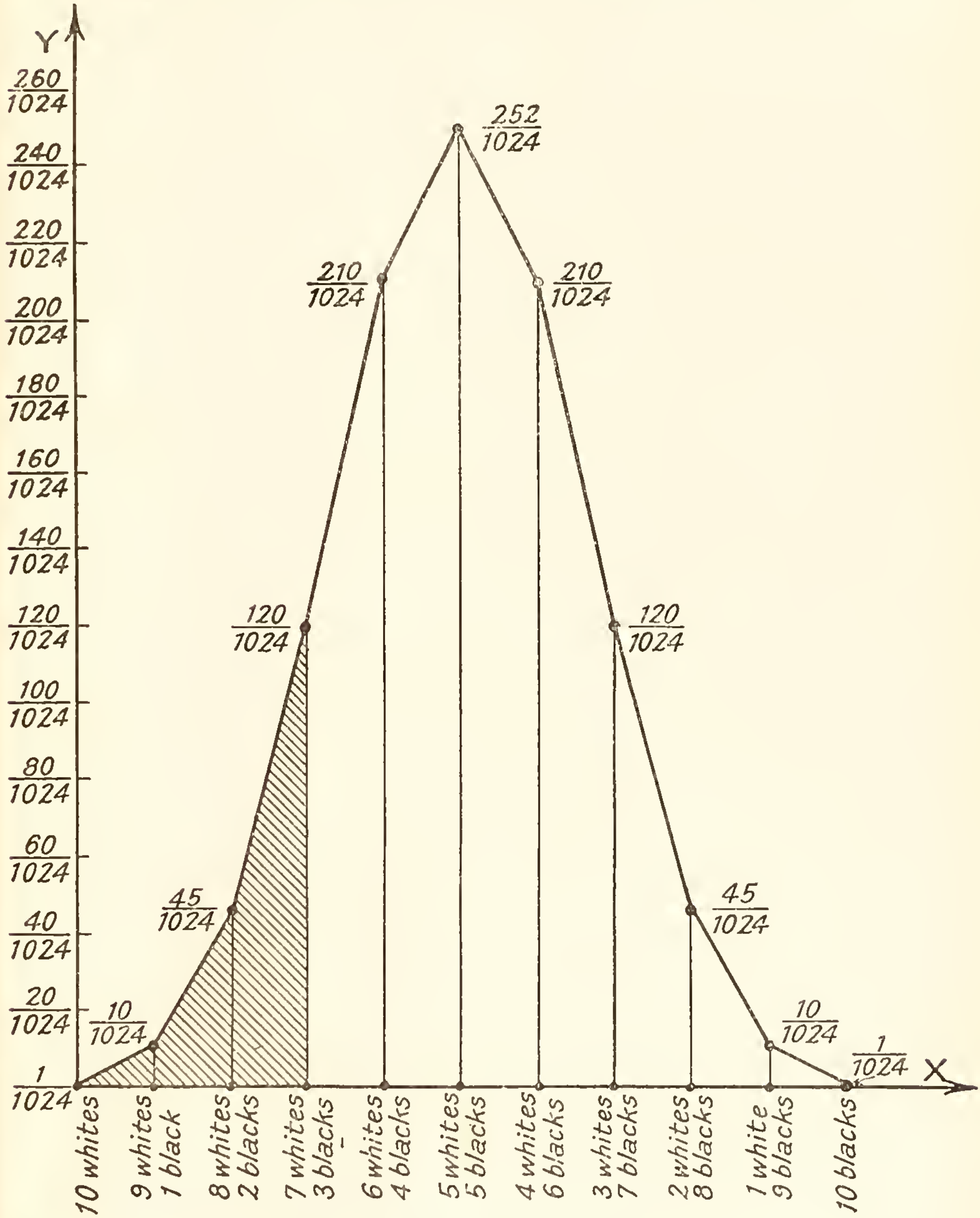


FIG. 148.—Graph for $(\frac{1}{2}+\frac{1}{2})^{10}$.

From the shaded portion of the polygon so drawn, as well as from the foregoing table, it will be seen that the probability

of the appearance of, say, seven *or more* whites in ten successive draws is the sum of the first four terms, viz.

$$0{\cdot}000977 + 0{\cdot}009766 + 0{\cdot}043945 + 0{\cdot}117187 = 0{\cdot}171875.$$

In other words, there is a chance of a little more than 1 in 6 of such an event happening.

Normal Frequency Curve.—If in the binomial $(\frac{1}{2}+\frac{1}{2})^n$, n is made larger and larger, the line resulting from connecting the tops of the ordinates drawn to represent the various terms in the expansion at equal intervals along the x axis will come to resemble more and more a smooth curve like the one in fig. 149. When n is infinitely large the curve becomes perfectly smooth and regular. Such a curve is often called a *normal frequency curve* or *normal probability curve*, and its importance to the biologist lies in the fact that **not only do errors of observation in experimental work usually correspond in frequency with the ordinates of such a curve** (so that small errors of observation occur with greater frequency than large errors, and positive and negative errors of the same magnitude occur with the same frequency), **but certain biological and anthropological statistics, such as heights and mental abilities of people, the sizes of their red blood cells, etc., are found to be adequately represented by curves of this type.** From the fact that the curve fits errors of observation, it is also called the *normal curve of error*, or the *Gauss-Laplace curve*.

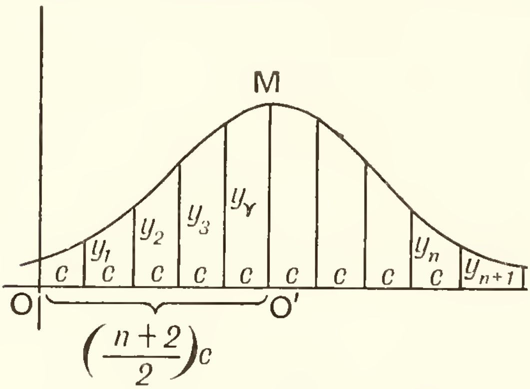


FIG. 149.—Normal Frequency Curve.

We shall return to the analysis of this curve on p. 406 *et seq.*

Any distribution that can be fitted with such a curve is called a *normal distribution*.

EXAMPLES.

(1) Out of 717,907 sets of twins Nichols found the sexes to be as follows: (1) Both males: 234,497; (2) one male, one female: 264,098; (3) both females: 219,312. How many of these sets were identical or uniovular twins?

Uniovular twins must necessarily be of like sex (M *or* F), while binovular or fraternal twins may be either of like (M *or* F) or unlike (M *and* F) sex.

Hence if we call the children in a pair of binovular twins (a) and (b), the following combinations are possible:—

$$\underbrace{\text{(i) (a) and (b) males;}}_{\text{One set, both males.}} \quad \underbrace{\text{(ii) (a) male, (b) female;} \quad \text{(iii) (a) female, (b) male;}}_{\text{Two sets of unlike sex.}}$$

$$\underbrace{\text{(iv) (a) and (b) females.}}_{\text{One set, both females.}}$$

Therefore in a large number of twin births the number of male, mixed and female pairs should be in the ratios 1 : 2 : 1, and the number of pairs of like (M *or* F) sex and unlike (M *and* F) sex should be in the ratio 1 : 1.

Hence for every 100,000 mixed pairs of binovular twins there should be no less and *no more* than 100,000 like pairs of such twins, so that any excess of like-sex pairs must necessarily be uniovular ones.

In Nichols' figures the unlike-sex pairs were 264,098, for which there should be an equal number of like-sex pairs. But the actual number of like-sex pairs was $234{,}497 + 219{,}312 = 453{,}809$.

$\therefore$ $453{,}809 - 264{,}098 = 189{,}711$ *pairs of like sex must have been uniovular.*

$$= \frac{189{,}711 \times 100}{717{,}907} = 26{\cdot}4 \text{ per cent. of all twins;}$$

$$\text{or } \frac{189{,}711 \times 100}{453{,}809} = 41{\cdot}8 \text{ per cent. of all like-sex twins.}$$

(2) Of 166 triplets, R. A. Fisher found the sexes to be as follows: All males: 30; 2 males, 1 female: 40; 1 male, 2 females: 65; All females: 31. How many of these were not fraternal or triovular triplets?

Calling the children of fraternal or triovular triplets (a), (b) and (c), the expectations out of every 8 sets are:—

(i) (a), (b) and (c) males = 1 set of "all males."

$$\left.\begin{array}{l}\text{(ii) (a) and (b) males, (c) female}\\ \text{(iii) (b) and (c) males, (a) female}\\ \text{(iv) (c) and (a) males, (b) female}\end{array}\right\} = 3 \text{ sets of "two males and one female."}$$

(v), (vi) and (vii) of similar combinations = 3 sets of "one male and two females."

(viii) (a), (b) and (c) females = 1 set of "all females."

So that the theoretical distribution of the sexes in a large collection of cases should be in the ratio of the terms of the binomial expansion

$$(\tfrac{1}{2} + \tfrac{1}{2})^3 = \tfrac{1}{8} + \tfrac{3}{8} + \tfrac{3}{8} + \tfrac{1}{8}.$$

Therefore the ratio of like-sex (M *or* F) to unlike-sex (M *and* F) sets should in triovular triplets be 1 : 3 (= 2 : 6).

In Fisher's collection the ratio is 61 : 105. But 105 triovular unlike-sex sets must have $105/3 = 35$ sets of triovular like-sex triplets.

Therefore $61 - 35 = 26$ *sets of like-sex triplets must have been other than triovular.*

$$\therefore \text{ Percentage of non-triovular sets } = \frac{26}{166} \times 100 = 15{\cdot}7 \text{ per cent. of all sets.}$$

$$= \frac{26}{61} \times 100 = 42{\cdot}6 \text{ per cent. of like-sex triplets.}$$

(3) The still-birth rate being 3 per cent., what should be the theoretical percentage of twins in which one is born dead and the other alive?

$$(0{\cdot}03+0{\cdot}97)^2 = \underbrace{0{\cdot}0009}_{\substack{\text{Chance of}\\ \text{both dead.}}} + \underbrace{2\times0{\cdot}03\times0{\cdot}97}_{\substack{\text{Chance of one dead}\\ \text{and one alive.}}} + \underbrace{0{\cdot}9409}_{\substack{\text{Chance of}\\ \text{both alive.}}}$$

∴ Chance of one only of twins being still-born $= 2\times0{\cdot}03\times0{\cdot}97 = 0{\cdot}0582$.

∴ Out of every 1000 binovular twin births one would expect some 58 such births in which one of the twins is born dead and the other born alive.

In uniovular twins one would theoretically expect no cases in which only one was alive or dead.

Statistical Constants—Averages and Dispersion.—In every statistical inquiry two kinds of numerical summaries must be used to give the main facts of the set of measurements under consideration, viz.:

(1) A number which represents the *type* of the group of measures or observations.

(2) A number which represents the *measure of the dispersion* or *scatter*, or degree of variability, of the measurements from the type.

Averages.—The number which represents the type of the group is called an *average*. Statistically, one speaks of three kinds of averages, viz. (*a*) the *common average* or *arithmetic mean*, (*b*) the *median*, and (*c*) the *mode*.

(*a*) The *Arithmetic Mean* (A.M.) or *Common Average* is obtained by dividing the sum of the individual measures by their number. Thus, if the group consists of n measures, $x_1, x_2, x_3, \ldots x_n$, then

$$\text{Arithmetic mean} = \frac{x_1+x_2+x_3+\ \ldots\ +x_n}{n} = \frac{\Sigma(x)}{n}.$$

The A.M. is usually indicated by $\bar{x}$.

(*b*) *The Median.*—If in any group of measurements the individual measurements be arranged in ascending order of magnitude, then the middlemost measurement, above and below which there is an equal number of measurements, is called the median. Thus, if 9 men of different heights or weights be arranged in a row in order of their height or weight, then the height or weight of the 5th man would form the median—because there would be an equal number of men on either side of the 5th. Generally, if there are $2n+1$ measurements ($2n+1$ representing any *odd* number), then the median is the $(n+1)^{th}$ measurement. If the number of measurements is even, *e.g.* 10 men, then the median is a height or weight intermediate between that of the 5th and the 6th man.

(*c*) *The Mode*, as its name implies, is the most frequent or fashionable measurement. Thus, if we have a table giving the heights of 43 boys—ranging from 53 to 63·75 inches, with intervals of 0·25 inch, and the largest number of boys was found to be 58·5 inches tall, then 58·5 would be the *mode*. In what follows we shall have to deal mostly with the arithmetic mean, but it is necessary that the student should be familiar with the meanings of the other two averages.

Dispersion or Degree of Variability.—There are several measures of variability, but we shall here deal with two only, viz. (*a*) *standard deviation* and (*b*) *coefficient of variation*, these being the ones of the greatest importance in the mathematical theory of statistics.

(*a*) *The Standard Deviation* (*S.D.* or σ) is the square root of the arithmetical average of the *squares* of the individual deviations from the mean, or, **the root mean square deviation from the mean.** Thus, if the deviations are d_1, d_2, d_3, . . . d_n, and the number of boys n, then

$$\text{S.D. or } \sigma = \sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + \ . \ . \ . \ + d_n^2}{n}},$$

$$= \sqrt{\frac{\Sigma d^2}{n}}.$$

(Where Σd^2 means the "sum of such terms as d^2.")

The value $\Sigma d^2/n$, *i.e.* σ^2, is known as the *variance*.

Two sets of measurements may have the same mean and yet be different in character. Thus in one group of five examination candidates the marks received were 28, 22, 27, 23 and 30 respectively, and in another group of five candidates the individual marks were 15, 25, 36, 19 and 35 respectively. The average number of marks received by each group was the same, viz. 26, but while in the first group all the candidates were near the average, the candidates in the second group consisted of some who were much above and others who were much below the average. The standard deviation measures the degree of scatter, or dispersion, about the mean of the candidates in each group. In the first group

$$\sigma = \sqrt{\frac{(28-26)^2 + (22-26)^2 + \ . \ . \ . \ + (30-26)^2}{5}}$$

$$= \sqrt{\frac{4 + 16 + 1 + 9 + 16}{5}} = \pm 3{\cdot}03.$$

In the second group

$$\sigma = \sqrt{\frac{(15-26)^2+(25-26)^2+\ .\ .\ .\ +(35-26)^2}{5}} = \pm\ 8{\cdot}85.$$

The difference between the two standard deviations may mean *either* that the two groups of candidates were really of markedly different degrees of ability, *or* that the precision of the examiner of the second group in awarding his marks was less than that of the examiner of the first group. Hence the standard deviation also measures the precision of an observer.

Bernoulli's Formula for the Standard Deviation.—It will be shown subsequently that if we have a group of n cases amongst which the probability of a certain event happening is p and that of the event not happening is q, then by Bernoulli's theorem $\sigma = \sqrt{npq}$.

If *rates* rather than *absolute* frequencies are considered, p becomes p/n and q becomes q/n. We then get

$$\sigma = \sqrt{n\cdot\frac{p}{n}\cdot\frac{q}{n}} = \sqrt{\frac{pq}{n}}\ \text{(see Example (2), p. 419).}$$

Thus, if the death-rate among a certain class of people is 147 per 1000, then the probability of death in that group is 0·147, and that of survival is $1-0{\cdot}147 = 0{\cdot}853$, so that the standard deviation is $\sqrt{1000\times0{\cdot}147\times0{\cdot}853} = \pm 11{\cdot}19$.

(*b*) *The Coefficient of Variation* (*C. of V.*) is the standard deviation σ expressed as a percentage of the mean M,

$$i.e.\ \text{C. of V.} = \frac{100\,\sigma}{\text{M}}$$

This coefficient of variation, therefore, measures the amount of the variability σ in terms of the mean.

The coefficient is used for measuring the relative variabilities of different characters which otherwise are not comparable. Thus if 924 people had pulse-rates varying between 45 and 116, with $\sigma = 11{\cdot}067$, whilst another 272 people had weights varying between 105 and 205 lbs. with $\sigma = 19{\cdot}95$, we cannot say whether pulse rate or weight is the more variable character, because we cannot compare pulse-beats with pounds weight. If, however, we know that the mean pulse-rate in the one group was 74·2, and the mean weight in the other group was 151·6 lbs., we have:

$$\text{C. of V. for pulse-rate} = \frac{11{\cdot}067\times100}{74{\cdot}2} = 14{\cdot}92 \text{ per cent. of the mean,}$$

and C. of V. for weight $= \dfrac{19{\cdot}95 \times 100}{151{\cdot}6} = 13{\cdot}16$ per cent. of the mean.

As the magnitude of the coefficient of variation is an abstract number totally independent of the unit of measure (pulse-rate or weight) employed, it enables us to see that although the standard deviation in the case of the pulse is less than that in the case of weight, pulse rate is really a slightly more variable character than weight.

Similarly, if a group of children of the same age had a mean height of, say, 41·3 inches, with $\sigma = 2{\cdot}24$, and a mean weight of 39·7 lbs., with $\sigma = 4{\cdot}4$ lbs., the C. of V. for height = 224/41·3 = 5·4, and C. of V. for weight = 440/39·7 = 11·1, showing that the weight of the children was about twice as variable as their height.

Weighted Averages.—We stated on p. 398 that the A.M. is found by dividing the sum of the individual measurements by their number. This is only true if all measurements are of equal importance or reliability, or, as we say, of equal "weight." If they are not of equal value or "weight," we must first "weight" each observation according to its importance or reliability—although it is not always easy to decide what "weight" to give to any particular measurement. The following examples will elucidate the meaning of "weighting."

The first two columns of the table below show the frequency distribution of the duration of pregnancy in 250 women. What is the average duration of pregnancy?

(1) Duration in Days (x).	(2) Frequency (f).	(3) Frequency Multiplied by Respective Duration (fx).	(4) Deviation of Duration from Mean (d).	(5) fd^2.
190	1	190	−91·68	$1 \times 91{\cdot}68^2$
210	1	210	−71·68	$1 \times 71{\cdot}68^2$
240	1	240	−41·68	$1 \times 41{\cdot}68^2$
250	2	500	−31·68	$2 \times 31{\cdot}68^2$
260	22	5720	−21·68	$22 \times 21{\cdot}68^2$
270	38	10260	−11·68	$38 \times 11{\cdot}68^2$
280	79	22120	− 1·68	$79 \times 1{\cdot}68^2$
290	78	22620	+ 8·32	$78 \times 8{\cdot}32^2$
300	16	4800	+18·32	$16 \times 18{\cdot}32^2$
310	9	2790	+28·32	$9 \times 28{\cdot}32^2$
320	2	640	+38·32	$2 \times 38{\cdot}32^2$
330	1	330	+48·32	$1 \times 48{\cdot}32^2$
Totals	$250 = \Sigma f = n$	$70420 = \Sigma fx$	..	$56293{\cdot}8 = \Sigma fd^2$

Clearly the various magnitudes 190, 250, 260, etc., are not of equal importance, because while some are derived from no more than one observation, others are based on many, such as 22, 79, etc., observations. Hence each magnitude must, in order to receive its due value, be "weighted" by multiplying it by its frequency as shown in column (3). The total in that column, 70420, divided by the total number of cases, 250, gives the A.M., viz. 281·68 days.

Hence A.M. $= \dfrac{\Sigma fx}{n}$, where f is the frequency or "weight" of any particular observation.

Column (5) of the table also gives the square of the deviation of each magnitude from the mean multiplied by the frequency, viz. fd^2. The sum of these products, viz. Σfd^2, is 56293·8, as shown at the bottom of the column.

$$\therefore \quad \text{S.D.} = \sqrt{\frac{\Sigma fd^2}{n}} = \sqrt{\frac{56293 \cdot 8}{250}} = 15 \cdot 01.$$

Simplified Method of finding the A.M. and S.D.—It is clear from columns (3) and (5) of the previous example that the arithmetical labour of computing the A.M. and especially the S.D. may be very heavy. This may, however, be very much reduced by adopting a couple of simple devices.

(1) *A.M.*—A glance at columns (1) and (2) of the table shows at once that the bulk of the observations concentrates round the figure 280. Hence the first simplification is to *change the origin* by taking 280 as a provisional or *arbitrary mean* and record the observations as measured from 280 as the origin, as follows:—

$$-90 \ (i.e.\ 190 = 280-90), \ -70, \ . \ . \ . \ +40, \ +50 \ (i.e.\ 330 = 280+50).$$

Column (3) of the table will then become modified into the simpler form

$$[-90, \ -70, \ -40, \ -60 \ (= -30\times 2), \ -440, \ -380, \ 0, \ +780, \ +320, \ +270, \ +80, \ +50]$$

whose algebraic sum $= 420$.

$$\therefore \quad \text{A.M. as measured from 280 as origin} = \frac{420}{250} = 1 \cdot 68 \text{ days.}$$

$\therefore$ *True A.M.* as measured from the actual zero $= 280 + 1{\cdot}68$ $= 281{\cdot}68$ days.

The second simplification consists in changing the unit, by making the group interval (viz. 10 days) the unit, when -90, -70, etc., become -9, -7, etc., as shown in the 2nd column of the next table. The 4th column of the same table shows each simplified value of fx, and the number at the bottom of the column shows that the modified $\Sigma fx = 42$ units $= 420$ days, so that

$$\textit{True A.M.} = 280 + \frac{42 \times 10}{250} = 281{\cdot}68 \text{ days.}$$

Note.—The algebraic sum, Σfx, of the products of the deviations from the origin by the respective frequencies is called the **first moment** of the distribution with reference to that origin, and the first moment divided by the total frequency, *i.e.* $\Sigma fx/n$, is called the **first moment coefficient.**

We therefore have the general formula

$$m = m_0 + p,$$

where m = true mean (281·68 days in our example),

m_0 = arbitrary mean (280 days in our example),

p = first moment coefficient about the arbitrary mean as origin $\left(\frac{42 \times 10}{250} = 1{\cdot}68 \text{ in our example}\right)$.

Corollary.—From this formula it follows that when $m_0 = m$, *i.e.* when the arbitrary mean coincides with the true mean, then $p = 0$. In other words, *the first moment of a distribution about the true mean is zero* (*i.e.* the sum of all the negative deviations from the true mean is equal to the sum of all the positive deviations). Thus in our example

$$-91{\cdot}68 \times 1 - 71{\cdot}68 \times 1 - 41{\cdot}68 \times 1 - 31{\cdot}68 \times 2 - 21{\cdot}68 \times 22 - 11{\cdot}68 \times 38 - 1{\cdot}68 \times 79$$

$$+ 8{\cdot}32 \times 78 + 18{\cdot}32 \times 16 + 28{\cdot}32 \times 9 + 38{\cdot}32 \times 2 + 48{\cdot}32 \times 1$$

$$= -1321{\cdot}92 + 1321{\cdot}92 = 0.$$

Table showing the arrangement of the work for finding the A.M. and S.D.:—

(1) Duration of Pregnancy in Days.	(2) Deviation (x_0) from Arbitrary Mean (280) in Terms of 10 as Unit.	(3) Frequency (f).	(4) fx_0.	(5) fx_0^2.
190	−9	1	−9	81
210	−7	1	−7	49
240	−4	1	−4	16
250	−3	2	−6	18
260	−2	22	−44	88
270	−1	38	−38	38
280	0	79	0	0
290	+1	78	+78	78
300	+2	16	+32	64
310	+3	9	+27	81
320	+4	2	+8	32
330	+5	1	+5	25
Totals		250 $= n$	150 − 108 $= 42$ $= \Sigma fx_0$	570 $= \Sigma fx_0^2$

(2) *S.D.*—Computation of the standard deviation (σ_0) of the distribution from a changed origin, such as the arbitrary mean, instead of from the true mean, very greatly reduces arithmetical labour. This, as can be seen from the bottom of column 5 of the foregoing table, is readily accomplished, since $\sigma_0^2 = \Sigma fx_0^2/n$ (x_0 stands for deviation from the arbitrary mean). The S.D. (σ) of the distribution from the true mean is then obtained by means of a very simple correction, viz. $\sigma = \sqrt{\sigma_0^2 - d^2}$, where d = difference between the true and arbitrary means. For:

Let x_0 be the deviation of any particular measurement from the arbitrary mean 280,

and x be the deviation of the same measurement from the true mean 281·68.

Then obviously $x_0 = x + 1{\cdot}68$, where 1·68 is the difference between the two means $= d$.

(Thus, in the case of the measurement 290, for instance, $x_0 = 10$ and $x = 8{\cdot}32$, $\therefore\ x_0 = x + 1{\cdot}68$.)

$$\therefore \quad x_0^2 = (x+1{\cdot}68)^2 = x^2 + 3{\cdot}36x + 1{\cdot}68^2,$$

$$\therefore \quad \Sigma f x_0^2 = \Sigma f x^2 + 3{\cdot}36 \Sigma f x + 1{\cdot}68^2 n$$
$$= \Sigma f x^2 + 1{\cdot}68^2 n$$

(since by corollary on p. 403 $\Sigma fx = 0$),

$$\therefore \quad \frac{\Sigma f x_0^2}{n} = \frac{\Sigma f x^2}{n} + 1{\cdot}68^2,$$

i.e.
$$\sigma_0^2 = \sigma^2 + 1{\cdot}68^2.$$

$$\therefore \quad \sigma = \sqrt{\sigma_0^2 - 1{\cdot}68^2}.$$

Column 5 of the table makes

$\sigma_0^2 = \dfrac{570}{250}$ square "group interval" units of 10 days $= \dfrac{57{,}000}{250}$ $= 228$ days.

$$\therefore \quad \sigma = \sqrt{228 - 1{\cdot}68^2} = \sqrt{228 - 2{\cdot}82} = \sqrt{225{\cdot}18} = 15{\cdot}01.$$

Thus if d = difference between the two means

$$\sigma = \sqrt{\sigma_0^2 - d^2}.$$

Definition.—In the same way as $\Sigma f x_0$ is called the first moment of the distribution about the arbitrary mean (x_0 being the deviation of any observation from that mean in terms of any convenient unit), so *$\Sigma f x_0^2$ is called the second moment, $\Sigma f x_0^3$ the third moment, and in general $\Sigma f x_0^n$ the nth moment* of the distribution about the arbitrary mean. *The various moments divided by Σf (or by n) are called the first, second, . . . nth moment coefficients.*

Note.—The assessment of the "weight" of any laboratory measurement by means of its frequency is valid only when all the measurements have been made with equal care and under equally favourable conditions. It is, however, a not uncommon laboratory experience that one single observation may be so good and accurate as to outweigh several others made under less favourable conditions. In that case it is impossible to express, with mathematical precision, the relative values of the different measurements, and the "weighting" must be left to the discretion of the observer, who is familiar with the circumstances under which each observation was made.

EXAMPLES.

(1) Twenty-five people are ranked in order of merit, viz. 1, 2, 3, . . . 23, 24, 25. What is the mean rank and the standard deviation?

The mean of the first n natural numbers is $\dfrac{(n+1)}{2}$, and their sum is $\dfrac{n(n+1)}{2}$ (see p. 69).

Also, the sum of the squares is $\frac{n(n+1)(2n+1)}{6}$ (see p. 71).

$\therefore$ If we take zero as the origin (*arbitrary* mean)

$$\sigma_0^2 = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}$$

and d (*i.e.* difference between true and arbitrary means) $= \frac{(n+1)}{2}$.

$$\therefore \quad \sigma = \sqrt{\sigma_0^2 - d^2} = \sqrt{\frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}} = \sqrt{\frac{n^2-1}{12}}.$$

Hence, when $n = 25$, the mean is $\frac{25+1}{2} = 13$, and the S.D. $= \sqrt{\frac{624}{12}}$ $= \sqrt{52} = 7{\cdot}211$.

Equation of Normal Frequency Curve.—On p. 396 we defined a normal frequency or probability curve as one of which the ordinates drawn at equal intervals along the x axis represent respectively the various terms of the binomial expansion $(\frac{1}{2}+\frac{1}{2})^n$ when n is infinitely large. Let the curve in fig. 150 be one of which the ordinates $y_1, y_2, y_3, \ldots y_n, y_{n+1}$ drawn at the abscissal points $x_1 = c, x_2 = 2c, x_3 = 3c, \ldots x_n = nc, x_{n+1} = (n+1)c$ respectively represent the terms

$$\left(\frac{1}{2}\right)^n, \frac{n}{2^n}, \frac{n(n-1)}{2^n \,.\, 2\,!}, \ldots \frac{n}{2^n}, \left(\frac{1}{2}\right)^n$$

of such an expansion.

(i) *Let us first take the case where n is large but finite:*—If we

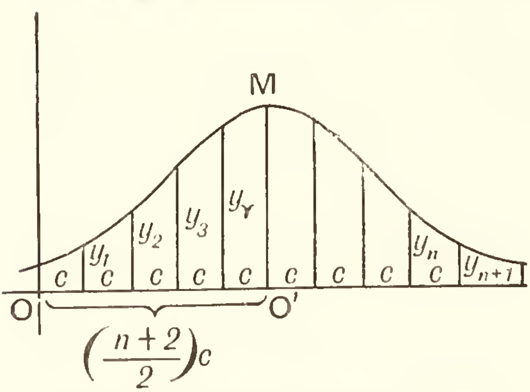


FIG. 150.—Normal Frequency Curve.

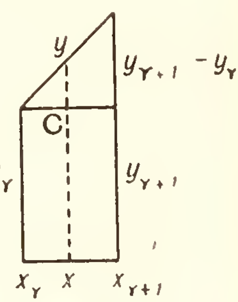


FIG. 151.

take any abscissa $x_r = rc$, its corresponding ordinate will be

$$y_r = \frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{1\,.\,2\,.\,3\ \ldots\ (r-1)} \cdot \frac{1}{2^n}.$$

Similarly the ordinate drawn at the next abscissal point $x_{r+1} = (r+1)c$, will be

$$y_{r+1} = \frac{n(n-1)(n-2)\ .\ .\ .\ (n-r+1)}{1\ .\ 2\ .\ 3\ .\ .\ .\ r} \cdot \frac{1}{2^n}.$$

Hence the slope of the curve between the ordinates y_r and y_{r+1} is (see fig. 151 and p. 161)

$$\frac{y_{r+1} - y_r}{c} = \frac{1}{c}y_r\left(\frac{n-r+1}{r} - 1\right) = \frac{y_r}{cr}(n - 2r + 1).$$

Now if n is very large and c correspondingly small, the slope of the tangent to the curve at the point (x, y) midway between (x_r, y_r) and (x_{r+1}, y_{r+1}) will tend to become identical with that between the ordinates y_r and y_{r+1}.

But $x = \frac{1}{2}(x_r + x_{r+1}) = \frac{1}{2}[rc + (r+1)c] = \frac{c}{2}(2r+1)$

and $\quad y = \frac{1}{2}(y_r + y_{r+1}) = \frac{1}{2}y_r\left(1 + \frac{n-r+1}{r}\right) = \frac{y_r}{2r}(n+1).$

Whence $\quad y_r = \dfrac{2ry}{(n+1)}.$

$\therefore$ Slope at point (x, y), which, as we have seen, is $\dfrac{y_r}{cr}(n - 2r + 1)$

$$= \frac{2ry}{cr(n+1)}(n-2r+1) = \frac{2y}{c(n+1)}\Big[(n+2) - (2r+1)\Big]$$

$$= \frac{2y}{c(n+1)}\left[(n+2) - \frac{2x}{c}\right], \text{ since } x = \frac{c}{2}(2r+1)$$

$$= \frac{4y}{c^2(n+1)}\left[\frac{(n+2)}{2}c - x\right].$$

But $\dfrac{(n+2)}{2}c$ is obviously the abscissa OO′ of the point O′ (the foot of the middle or maximum ordinate O′M) as measured from O as origin. (In fig. 150, for instance, where the last ordinate y_{n+1} is the 9th, $n = 8$ and $OO' = 5c = \dfrac{(8+2)}{2}c$.) Then, by *transferring the origin from* O *to* O′, $\left(\dfrac{n+2}{2}\right)c$ becomes zero and the slope at (x, y) becomes $\dfrac{-4yx}{c^2(n+1)}$.

As c is unit deviation on the abscissa axis, we can put $c = 1$; we then have

$$\text{Slope at point } (x, y) = -\frac{4yx}{(n+1)}.$$

(ii) *Now assume n to be infinitely large:*—The joins of the tops of the various ordinates $y_1, y_2, \ldots y_n, y_{n+1}$ will then form the smooth continuous normal frequency curve whose slope, $\frac{dy}{dx}$, at the point (x, y) is given by

$$\frac{dy}{dx} = -\frac{4yx}{(n+1)} = -\frac{4yx}{n} \text{ (since } n = \infty\text{)}.$$

As the equation contains no y_r or y_{r+1}, it may be taken as true for *all* points on the curve, and therefore is the differential equation of the curve with O′ as origin.

$$\therefore \quad \int\frac{dy}{y} = -\frac{4}{n}\int x dx$$

i.e. $\log_e y = -\frac{2x^2}{n} + \log_e \mathrm{A}$ (where $\log_e \mathrm{A}$ = integration constant).

$$\therefore \quad y = \mathrm{A}e^{-\frac{2x^2}{n}}$$

But, as we saw on p. 400, σ (the standard deviation of the distribution) $= \sqrt{npq}$, therefore $\sigma^2 = npq = n/4$ in the case of a normal distribution, where $p = q = 1/2$. We therefore have $n = 4\sigma^2$, and our equation for the normal frequency curve becomes

$$y = \mathrm{A}e^{-\frac{x^2}{2\sigma^2}}$$

y represents the frequency or probability of an error or deviation, $\pm x$, from the mean, *i.e.* from the point O′.

Evaluation of A.—The area enclosed by the whole curve between $x = \pm\infty$ (on the right and left of the origin O′) is given by $\int_{-\infty}^{+\infty} y dx = \mathrm{A}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}dx = \mathrm{A}\sigma\sqrt{2\pi}$ (see p. 311).

But the area enclosed by the whole curve also represents the total frequency or probability, which

$$= y_1 + y_2 + y_3 + \ldots \text{ to } \infty = (\tfrac{1}{2} + \tfrac{1}{2})^\infty = 1$$

$$\therefore \quad \mathrm{A}\sigma\sqrt{2\pi} = 1$$

$$\therefore \quad \mathrm{A} = \frac{1}{\sigma\sqrt{2\pi}}, \quad \left(= \frac{0{\cdot}3989}{\sigma}\right)$$

As A is the value of y when $x = 0$, $\left(e^{-\frac{x^2}{2\sigma^2}} \text{ being } e^0 = 1\right)$, it may be written as y_0, and represents the height of the middle or maximum ordinate O′M.

∴ **Final Equation of Normal Frequency or Probability Curve is**

$$y = y_0 e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Properties of the Normal Frequency Curve.—(1) The curve is symmetrical about the y axis because for any value of y there are two equal and opposite values of x (see note on p. 67). Thus, whether $x = \pm n$ the value of y is the same. This means that any two errors or deviations from the mean which are numerically equal but of opposite sign occur with the same frequency.

(2) It is entirely situated above the x axis, since $y = Ae^{-\frac{x^2}{2\sigma^2}}$ can never be < 0.

(3) The axis of x is an asymptote, *i.e.* the curve tails off to the right and left of the y axis but never touches the x axis, for as x tends to $\pm \infty$, y tends to become 0 but never reaches it.

(4) The value of the maximum ordinate O′M is, as we have seen,

$$\frac{1}{\sigma\sqrt{2\pi}}, \quad \left(= \frac{0{\cdot}3989}{\sigma}\right)$$

and occurs where $x = 0$. The ordinates get smaller and smaller on either side of the mean as x, or the magnitude of the positive or negative error, increases. This means that the larger the error (whether positive or negative) the less likely it is to occur (see pp. 412 and 413).

(5) The point P_1 on the curve, whose abscissa $= \sigma$ (fig. 153), is a point of inflexion, *i.e.* where the curve changes from concave to convex. This has been proved on p. 227, Example 2. Hence the position of the point of inflexion P_1 depends upon the value of σ and therefore also upon the value of the maximum ordinate.

(6) The height of the ordinate N_1P_1 (fig. 153) is given by the equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} \text{ (since } x = +\sigma)$$

$$= \frac{0{\cdot}607}{\sigma\sqrt{2\pi}} = 0{\cdot}607 \text{ of the maximum ordinate (see (4))} = \frac{0{\cdot}2420}{\sigma}$$

Error and Probable Error.—The deviation of any particular observation (out of a series of observations) from the mean is

termed the **error** of the observation. If these deviations, or errors, are symmetrically distributed on either side of the mean then they can be fitted by a normal probability curve. Such a curve is often called the *normal curve of error* or *normal frequency curve*, and has as its equation

$$y = \frac{N}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

where N = number of observations,
σ = standard deviation of the errors,
and y = the frequency of any deviation or error whose magnitude is $\pm x$.

If the total frequency be called unity, then the equation becomes

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

where y now represents the probability of a deviation of magnitude $\pm x$, the probability of that deviation being its frequency divided by the total frequency N of all deviations.

The *probable error* (*p.e.*) is that value ($\pm r$) of the abscissa on either side of the mean (X and X_1, fig. 152) such that the ordinates XN, X_1N_1 enclose half the area enclosed by the whole curve, so

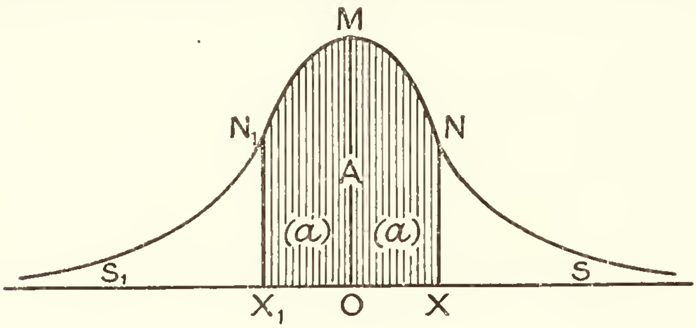


FIG. 152.

that there is an even chance that the true value of the quantity to be determined lies between these limits. Thus the period of gestation has been found to be 282·5 ± 0·55 days (± 0·55 day being the probable error). This means that half of all pregnancies last between (282·5 + 0·55) and (282·5 − 0·55) days, *i.e.* between 283·05 and 281·95 days, the other half lasting either less than 281·95 days or more than 283·05 days.

If, for the probability curve (fig. 152) OX or OX_1 = the

probable error, then by definition the ordinates XN, X_1N_1 bisect each half of the curve on either side of the mean OM, so that area (A) of portion X_1N_1MNX = half total area enclosed by probability curve.

Note.—The term "probable error" is somewhat confusing, since the error in question is not by any means the most probable, and as Whipple suggests, "median deviation" is much more descriptive of its character.

Formula for Magnitude of Probable Error in Terms of Standard Deviation.—Since, by definition, the chance of an error falling within the limits $\pm r$ is exactly equal to the chance of an error falling outside these limits, and since the chance of the occurrence of an error of any undefined magnitude (*i.e.* within the limits $\pm \infty$) is geometrically expressed by the total area of the curve of error, *i.e.* by 1,

$\therefore$ Area A covered by the portion of the curve representing the frequency of errors between the limits of $\pm r$ (*i.e.* of the probable error) must equal half the total area enclosed by the whole probability curve, $= \frac{1}{2}$.

$$i.e. \quad \frac{1}{\sigma\sqrt{2\pi}}\int_{-r}^{+r} e^{-\frac{x^2}{2\sigma^2}}dx = \tfrac{1}{2}$$

$$\therefore \quad \frac{1}{\sigma\sqrt{2\pi}}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}dx = \tfrac{1}{4}$$

$$\text{or} \quad \frac{1}{\sigma}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}dx = \frac{\sqrt{2\pi}}{4} = 0{\cdot}6265.$$

Evaluating the integral by expanding $e^{-\frac{x^2}{2\sigma^2}}$ into a series, we have

$$\text{since} \qquad e^{-\frac{x^2}{2\sigma^2}} = 1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{2\,!\,4\sigma^4} - \frac{x^6}{3\,!\,8\sigma^6} + \ldots \quad \text{(p. 80)}$$

$$\therefore \quad \frac{1}{\sigma}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{\sigma}\int_{0}^{+r}\left[1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \ldots\right]dx$$

$$= \left[\frac{x}{\sigma} - \frac{x^3}{6\sigma^3} + \frac{x^5}{40\sigma^5} - \frac{x^7}{336\sigma^7} + \ldots\right]_0^r$$

$$\therefore \qquad 0{\cdot}6265 = \left(\frac{r}{\sigma} - \frac{r^3}{6\sigma^3} + \frac{r^5}{40\sigma^5} - \frac{r^7}{336\sigma^7} + \ldots\right)$$

$$\text{or} \qquad \frac{r}{\sigma} - \frac{r^3}{6\sigma^3} + \frac{r^5}{40\sigma^5} - \frac{r^7}{336\sigma^7} - 0{\cdot}6265 = 0$$

(the higher terms of r being negligible).

The equation is satisfied by $\frac{r}{\sigma} = 0{\cdot}6745$

$$\therefore \quad r = 0{\cdot}6745\sigma.$$

In other words, **the probable error is approximately 2/3 of the standard deviation or standard error.**

Let us now compute the area a (or A/2) when x/σ has any other value. For instance, when $x/\sigma = 1$ or $x = \sigma$, we have

$$a \text{ or } \frac{A}{2} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^{x=\sigma} e^{-\frac{x^2}{2\sigma^2}}dx$$

$$= 0{\cdot}3989\left[\frac{x}{\sigma} - \frac{x^3}{6\sigma^3} + \frac{x^5}{40\sigma^5} - \frac{x^7}{336\sigma^7} + \ldots\right]_{x=0}^{x=\sigma}$$

$$= 0{\cdot}3989(1 - \tfrac{1}{6} + \tfrac{1}{40} - \tfrac{1}{336} + \ldots)$$

$= 0{\cdot}3413$ (more exact value is $0{\cdot}34134$ of the total area of the curve).

In a similar way a table—called a **probability integral table**—has been computed giving the values of A/2 (and therefore of A), for different values of x/σ varying from 0·00 onwards at intervals of 0·01. Intermediate values are found by the method of "proportional parts."

a, or A/2, measures the total frequency of all deviations or errors between 0 and $\pm\, x/\sigma$, and A measures the total frequency of the errors between $-x/\sigma$ and $+x/\sigma$, *i.e.* between the limits of $\pm\, x/\sigma$ (see fig. 152).

Thus, for $x/\sigma = 1$, as we have just seen, $A/2 = 0{\cdot}34134$,

$$\therefore \quad A = 0{\cdot}6827.$$

for $x/\sigma = 2$ the probability integral table gives

$$A = 0{\cdot}9545$$

for $x/\sigma = 3$ the probability integral table gives

$$A = 0{\cdot}9973.$$

$(1 - A)$ in each case measures the frequency of all the errors which are numerically greater than $\pm\, x/\sigma$ (*i.e.* the portions of the curve to the right of XN and to the left of X_1N_1), and $A/(1 - A)$ gives the odds against the occurrence of an error numerically greater than $\pm\, x/\sigma$.

Thus for $\frac{x}{\sigma} = 1, \quad \frac{A}{1 - A} = \frac{0{\cdot}6827}{0{\cdot}3173} = \text{about } \frac{2}{1},$

for $\qquad \frac{x}{\sigma} = 2, \quad \frac{A}{1 - A} = \frac{0{\cdot}9545}{0{\cdot}045} = \frac{21}{1},$

and for $\frac{x}{\sigma} = 3$ the ratio is $\frac{0 \cdot 9973}{0 \cdot 0027} = \frac{370}{1}$,
i.e. the odds against an error numerically greater than $\pm 3\sigma$ *are* 370 : 1.

Also the area of the portion of the curve on the right of XN is

$$0 \cdot 5 - \frac{A}{2} = \frac{1 - A}{2}$$

and the area of the portion of the curve on the left of XN or on the right of X_1N_1 is

$$0 \cdot 5 + \frac{A}{2} = \frac{1 + A}{2}.$$

EXAMPLE.

Verify graphically the odds against errors numerically greater than $\pm\sigma$, $\pm 2\sigma$ and $\pm 3\sigma$ in the case of a normal frequency distribution of 95 observations with $\sigma = 5$.

Substituting the numerical values for N and σ we have:

Equation of required curve is $y = \frac{95}{5\sqrt{2\pi}} e^{-\frac{x^2}{50}} = 7 \cdot 58 e^{-\frac{x^2}{50}}$

$\therefore$ When $x = 0$, $y = 7 \cdot 58$ = maximum ordinate OB (fig. 153).
,, $x = 5 (= \sigma)$, $y = 7 \cdot 58 e^{-\frac{1}{2}} = 7 \cdot 58 \times 0 \cdot 607 = 4 \cdot 60$ = ordinate N_1P_1
,, $x = 10 (= 2\sigma)$, $y = 7 \cdot 58 e^{-2} = 7 \cdot 58 \times 0 \cdot 135 = 1 \cdot 02$ = ,, N_2P_2
,, $x = 15 (= 3\sigma)$, $y = 7 \cdot 58 e^{-4 \cdot 5} = 7 \cdot 58 \times 0 \cdot 011 = 0 \cdot 08$ = ,, N_3P_3.

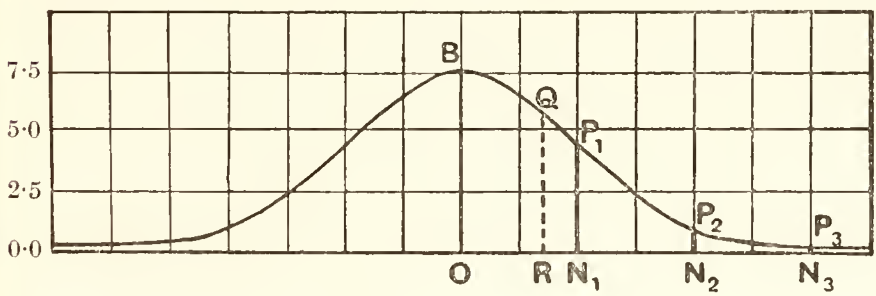


FIG. 153.—Curve of $y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ for $\sigma = 5$ and N = 95

$$= \frac{95}{5\sqrt{2\pi}} e^{-\frac{x^2}{50}} = 7 \cdot 58 e^{-\frac{x^2}{50}}$$

(7·58 being the height of the maximum ordinate OB, *i.e.* when $x = 0$).

Also, when $x = 0 \cdot 6745\sigma = 3 \cdot 3725$ = the probable error r, y (*i.e.* ordinate RQ) $= 7 \cdot 58 e^{-0 \cdot 2273} = 7 \cdot 58 \times 0 \cdot 7968 = 6 \cdot 04$.

The smooth curve drawn through the tops of these ordinates and continued symmetrically on the left of OB is shown in fig. 153. By re-drawing

it on a scale about five times as large and counting the squares (to the nearest half-square) included within the whole graph, as well as within the portions OBP_1N_1, OBP_2N_2 and OBP_3N_3, the student will obtain the required verification. He will see, for instance, that the area of the portion lying to the right of N_3P_3 is negligible, being about 1/370 of the area enclosed by half the curve. Similarly the probability of an error greater than 2σ or $3r$ (r being the probable error, *i.e.* about $2\sigma/3$), being due to chance alone is represented by the area of the portion of the curve lying to the right of N_2P_2, which is about 1/20 or 0·05 of half the curve.

It is very difficult to say how big a deviation must be in order to rule it out as being a chance error, but one for which x/σ is less than 2, *i.e.* one which may occur more often than once in 20 times, is by common convention considered as *insignificant*, *i.e.* as likely to occur as an error of random sampling, *i.e.* by chance alone.

Practical Use of the Frequency Curve and the Probability Integral Table.—It will be seen that in order to use the frequency curve for computing the frequency of errors or deviations lying between two given limits x_1 and x_2 one must first express those limits in terms of σ (*i.e.* as x_1/σ and x_2/σ), σ **being the unit of measurement in all statistical calculations,** and then find from the tables the values of the portions a of the curve for each of these values of x/σ or ξ, *i.e.* the values of the integral

$$\frac{1}{\sqrt{2\pi}}\int_0^{\xi} e^{-\xi^2/2}d\xi \text{ for } \xi = x_1/\sigma \text{ and for } \xi = x_2/\sigma.$$

These will give the frequencies of the errors or deviations from the mean within the ranges 0 to x_1/σ and 0 to x_2/σ. The total frequency of all the errors lying between x_1/σ and x_2/σ will therefore be given by the difference of these two values of the integral. The frequency of any *one* error $\xi = x_n/\sigma$ is

$$\frac{N}{\sigma\sqrt{2\pi}}e^{-\frac{x_n^2}{2\sigma^2}}$$

i.e. is represented by the length of the ordinate of the point x_n. The following examples will make this clear.

EXAMPLES.

(1) Out of 514 examination candidates, 5 obtained marks between 0 and 5, 9 obtained marks between 6 and 10, and so on for various "5 marks groups" up to 61–65. The mean of the marks was 31·93, and σ was 11·56 marks. What was the theoretical frequency of candidates with marks between 46 and 50?

The class interval 46–50 begins with 45·5 and ends with 50·5 (p. 4). As the deviation of 45·5 from the mean (31·93) is 13·57, therefore $x_{45\cdot5}/\sigma$ = 13·57/11·56 = 1·174.

∴ Frequency of candidates with marks between the mean and 45·5 is

given by $\frac{1}{\sqrt{2\pi}}\int_0^{1\cdot174} e^{-\xi^2/2}d\xi$. The table will give the value of this integral for $\xi = 1\cdot17$ as $0\cdot379$ and for $\xi = 1\cdot18$ as $0\cdot381$.

Therefore, by the method of "proportional parts," the value of the integral for $\xi = 1\cdot174$ is $0\cdot380$ of the total frequency, *i.e.* $0\cdot38 \times 514$.

Similarly, $$\frac{x_{50\cdot5}}{\sigma} = \frac{50\cdot5 - 31\cdot93}{11\cdot56} = 1\cdot606.$$

As the values of the integral for $\xi = 1\cdot60$ and $\xi = 1\cdot61$ are $0\cdot4452$ and $0\cdot4463$, therefore the value for $\xi = 1\cdot606$ is $0\cdot446$.

$\therefore$ Frequency of candidates with marks between the mean and $50\cdot5$ is $0\cdot446 \times 514$.

$\therefore$ Frequency of candidates with marks in the group interval 46–50 is $514 \times (0\cdot446 - 0\cdot380) = 34$. Actually, the observed frequency was 37.

(2) Taking the mean human stature for the British Isles as 67·46 inches, the mean for Cambridge students as 68·85 inches, and the common S.D. as 2·56 inches, what percentage of Cambridge students exceed the British mean in stature, assuming the distribution to be normal? (Udny Yule.)

If ON (fig. 154) = ordinate at mean, *i.e.* at 68·85 inches, and PQ =

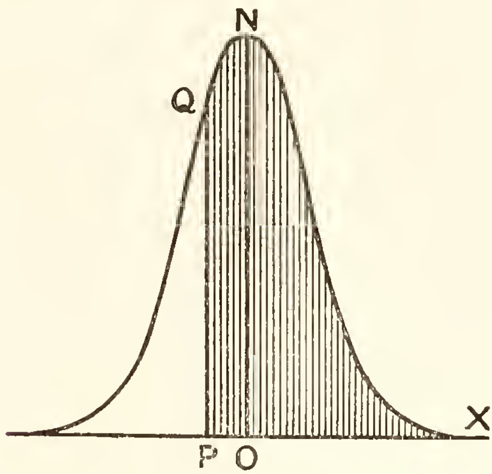


FIG. 154.

ordinate at 67·46 inches (*i.e.* the mean for the British Isles), then the area of the curve between PQ and the extreme right limit, *i.e.* the area PQNX, represents the frequency of students whose height exceeds 67·46 inches.

Now, total area = 1, $\therefore$ area ONX = 0·5.

Hence we have to calculate only the area of the portion PQNO. If this area $= a$, then $a + 0\cdot5$ = frequency required.

Now abscissa $\qquad OP = 67\cdot46 - 68\cdot85 = -1\cdot39.$

$$\therefore \quad \xi = \frac{OP}{\sigma} = -0\cdot54.$$

$\therefore$ $a = 0\cdot2054$ (from the probability integral table).

$\therefore$ Area of PQNX = 0·7054, which makes the required percentage 70·5.

(3) The death-rate of all persons between the ages of 25 and 45 is 147 per 1000, but out of every 1000 people of a certain class 185 died during that age interval in the same year. Can one say confidently that people of that class are more liable to death during that age interval? What are the maximum and minimum death-rates for that age interval that may occur as the result of the operation of chance alone?

The probability of death (p) between 25 and 45 years of age for all classes is 0·147.

∴ The probability of living (q) between 25 and 45 years of age for all classes is 0·853.

∴ S.D. $= \sqrt{1000 \times 0{\cdot}147 \times 0{\cdot}853} = 11{\cdot}19$, and therefore the difference between the actual and expected death-rates in the specified group is $(185 - 147)/11{\cdot}19 = 3{\cdot}4$ times the S.D. Such a deviation would, by chance alone, occur about 3 times in 10,000 years. Hence, one can say with great confidence that people of that particular class are particularly liable to die during the specified age interval.

The maximum positive or negative deviation which is considered as statistically insignificant is $3r$ (where r is the probable error) $= 3 \times 0{\cdot}6745\sigma = 23$ per 1000, so that the maximum and minimum death-rates to be expected by chance alone are 124 and 170 per 1000 respectively.

Note.—Although, strictly speaking, the integral table can only be used in cases of normal distributions, *i.e.* where $p = q = \frac{1}{2}$, nevertheless if n is large and p/q is not less than about 1/9, the result obtained is sufficiently accurate for most purposes.

(4) With a sex-ratio of 104 : 100 how many times in a century would one expect the birth of 383,000 or more males (*i.e.* a sex-ratio of 104·4 or more), and 381,500 or less males (*i.e.* a sex-ratio of 103·5 or less) respectively in one year, in England and Wales, on the assumption that the total annual number of births remains stationary at about 750,000?

A sex-ratio of 104 : 100 means 51 males to 49 females per 100 births.

∴ In the binomial $(p+q)^n$, $p = 0{\cdot}51$, $q = 0{\cdot}49$ and $n = 750{,}000$. Hence although p and q are unequal, the distribution may, owing to the

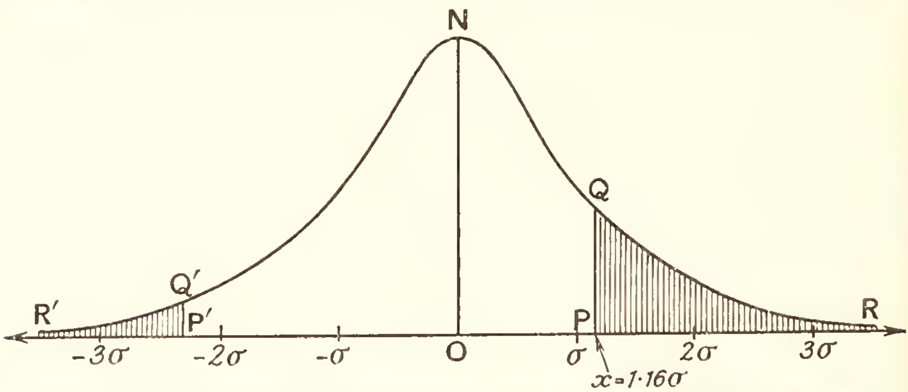


Fig. 155.—Indicating Probability of Birth of Given Number of Males in Any One Year.

large size of n, be represented by a normal frequency curve and the equation $y = y_0 e^{-\frac{x^2}{2\sigma^2}}$ applies, where $\sqrt{npq} = 433$ ($= \sigma$).

As the mean annual number of male births = $750{,}000 \times 0{\cdot}51 = 382{,}500$, therefore in fig. 155 the origin O represents 382,500, and the abscissa x of the point P, representing 383,000 males, is situated at a distance of 500 to the right of O.

$$\therefore \quad \frac{x}{\sigma} = \frac{500}{433} = 1{\cdot}16, \text{ which makes the area ONQP} = 0{\cdot}3770.$$

$\therefore$ Shaded area PQR, representing the sum of all the possible frequencies of 383,000 or more male births, is 0·123 of the total area of the curve (which, taken as unity, represents all the various possible frequencies from 0 to 750,000 males).

$\therefore$ The chances are that about three times in twenty-five years there will be 383,000 or more males born in one year.

Similarly if P′ represents 381,500 males, the abscissa of P′ is

$$-\frac{x}{\sigma} = \frac{1000}{433} = -2{\cdot}31.$$

$\therefore$ Area P′Q′R′ = $0{\cdot}5 - 0{\cdot}4896 = 0{\cdot}0104$, *i.e.* the chance of there being 381,500 or less males born in one year is about once in a century.

Note.—The student will observe that all the problems we have considered are really analogous to the one discussed in connection with the drawing of a ball (p. 394). We there saw that the probability of the appearance of, say, 7 or more white balls in 10 successive draws is the sum of the first four terms of the expansion $(p+q)^{10}$, viz. up to the term containing p^7q^3. Similarly, the total frequency of, say, 383,000 or more males could be obtained by adding up the terms in the expansion $(0{\cdot}51+0{\cdot}49)^{750{,}000}$ up to the term containing $(0{\cdot}51)^{383{,}000}(0{\cdot}49)^{367{,}000}$—a task which would be almost beyond human endurance. By means of the frequency curve, however, the problem is solved elegantly and most expeditiously.

The Relation between the Probability of the Occurrence of a Deviation of Given Magnitude x/σ and the Size of n (the Total Frequency of the Varieties in the Sample).—Since in the computation of frequencies of deviations of given magnitude (or lying between two given magnitudes) the deviations are measured in terms of σ (*i.e.* as x/σ), which as we have seen (p. 400) varies as $\sqrt{n}$, it is clear that the smaller the size of n, the smaller also will be the size of σ and therefore the greater will be the value of x/σ, thus making the area ONQP (in fig. 155) greater, and that of PQR, as well as of the ordinate PQ, correspondingly less, although the value of p (and therefore also of q) remains unaltered. The following is a numerical illustration of this.

EXAMPLE.

We saw that the probable frequency of the occurrence of a sex-ratio no greater than 103·5/100 is for the whole of England and Wales (where $n = 750{,}000$) no more than about once in a century. Let us see what is the probable frequency of such an event for a single town in which n, the

total mean annual number of births, is 7500, assuming the mean sex-ratio to be the same as for the whole of England and Wales, viz. 104/100, making $p = 0{\cdot}51$ and $q = 0{\cdot}49$.

Here $\sigma = \sqrt{npq} = 43{\cdot}3$, and the mean annual number of male births $= 7500 \times 0{\cdot}51 = 3825$. A sex-ratio of 103·5/100 in 7500 births corresponds to 3815 male births.

$$\therefore \quad \text{Deviation of } P' = \frac{3825 - 3815}{\sigma} = \frac{10}{43{\cdot}3} = 0{\cdot}23.$$

The probability integral table gives the value of ONQ′P′ for $x/\sigma = 0{\cdot}23$ as 0·091, whence area of P′Q′R′ = 0·409 = 40·9 per cent. of total area of curve.

This means that in the town in question a sex-ratio of 103·5/100 or less will occur about twice in five years, while in the whole of England and Wales such an event will only occur once in a century.

The Probable Error of the Sum or Difference of Two Unrelated Quantities.—Consider the following problem. The mean infant mortality rates in a given area during the 3 years 1917–1919 were: 104·23 ± 1·75; 99·24 ± 1·32; 111·61 ± 1·66. Are these fluctuations merely fortuitous or are they due to some environmental differences? We have seen that the significance of any statistical measurement is assessed by the number of times it exceeds the probable error, and hence whether, say, the difference between the infant mortality rates of the years 1917 and 1918, viz. 4·99 (= 104·23 − 99·24), is merely due to chance or to environmental influence must depend on the size of the probable error of that difference. But what is the probable error of the difference between the two mean infant mortality rates? The answer is given by the following *theorem*, the proof of which will be given later (p. 459):—

The probable error R of the sum or difference of two unrelated quantities is equal to the square root of the sum of the squares of the respective probable errors r_1 and r_2; *i.e.*

$$\mathbf{R} = \sqrt{r_1^2 + r_2^2} = 0{\cdot}6745\sqrt{\sigma_1^2 + \sigma_2^2}$$

Hence in the case we are considering we have difference between the infant mortality rates in 1917 and 1918 = 4·99.

r_1, the p.e. of the mean rate for 1917 = 1·75,

r_2, " " " 1918 = 1·32.

$$\therefore \quad R = \sqrt{1{\cdot}75^2 + 1{\cdot}32^2} = \pm\, 2{\cdot}19.$$

As 4·99/2·19 is less than 3, it is generally considered to be insignificant (p. 412).

On the other hand, difference between the 1918 and 1919 rates = 111·61 − 99·24 = 12·37.

And probable error of difference $= \sqrt{1{\cdot}66^2 + 1{\cdot}32^2} = \pm 2{\cdot}12$.

As $12{\cdot}37/2{\cdot}12 = 6$ (approximately), the difference in the two mortality rates cannot be due to chance alone, and must therefore be due to environmental differences.

EXAMPLES.

(1) The mean birth-weight of infants as calculated from the Report of the Anthropological Committee of the British Association in 1863 is:

Males, $3{\cdot}230 \pm 0{\cdot}016$ kgrm.; females, $3{\cdot}151 \pm 0{\cdot}015$ kgrm.

For the purpose of gaseous metabolism experiments in infants, a number of new-born babies were taken whose mean weights were:

Males, $3{\cdot}459 \pm 0{\cdot}0430$ kgrm.; females, $3{\cdot}336 \pm 0{\cdot}0564$ kgrm.

Are the latter infants a fair random sample of normal infants as judged by the B.A. Standard?

For the males, difference $= 3{\cdot}459 - 3{\cdot}230 \pm \sqrt{(0{\cdot}043)^2 + (0{\cdot}016)^2}$
$= 0{\cdot}229 \pm 0{\cdot}046$.

For the females, difference $= 3{\cdot}336 - 3{\cdot}151 \pm \sqrt{(0{\cdot}0564)^2 + (0{\cdot}015)^2}$
$= 0{\cdot}185 \pm 0{\cdot}058$.

Therefore both the male and female infants are significantly heavier than normal.

(2) Three groups of patients of equal sex distribution, etc., were treated for the same infectious disease by three different methods with the following results:—

	A.	B.	C.
Number of cases . . .	1200	865	1236
Number of deaths . . .	42	20	10

Do these figures show any advantage of one method over either of the others?

The percentage fatality rates are A, 3·5; B, 2·3; C, 0·81.

$$\therefore \sigma_A = \sqrt{\frac{1200 \times 3{\cdot}5 \times 96{\cdot}5}{1200 \times 100 \times 1200 \times 100}} = \sqrt{\frac{3{\cdot}5 \times 96{\cdot}5}{1200}} \text{ per cent.} = \sqrt{0{\cdot}281} \text{ per cent.}$$

$$\sigma_B = \sqrt{\frac{2{\cdot}3 \times 97{\cdot}7}{865}} \quad ,, \quad = \sqrt{0{\cdot}259} \quad ,,$$

$$\sigma_C = \sqrt{\frac{0{\cdot}81 \times 99{\cdot}19}{1236}} \quad ,, \quad = \sqrt{0{\cdot}065} \quad ,,$$

$\therefore$ In the case of groups A and B,

Difference in fatality rate $= 3{\cdot}5 - 2{\cdot}3 \pm \frac{2}{3}\sqrt{0{\cdot}281 + 0{\cdot}259} = 1{\cdot}2 \pm 0{\cdot}5$ per cent., which is not significant (since 1·2 is less than $3 \times 0{\cdot}5$).

In the case of groups A and C,

Difference in fatality rate $= 3{\cdot}5 - 0{\cdot}81 \pm \frac{2}{3}\sqrt{0{\cdot}281 + 0{\cdot}065} = 2{\cdot}7 \pm 0{\cdot}4$ per cent. As 2·7 is nearly 7 times 0·4, this difference is decidedly significant.

From this example we see that when *rates* rather than absolute frequencies are considered σ becomes $\frac{\sqrt{npq}}{n} = \sqrt{\frac{pq}{n}}$.

EXERCISES.

(1) The infant mortality rates for England and Wales in the years 1920 and 1921 were 80 and 83 per 1000 live births respectively, the total births being (in round numbers) 958,000 and 849,000 respectively. Is the rise in 1921 fortuitous or significant?

$$\left[\textit{Answer}, \text{ Difference } = 3 \pm \frac{2}{3}\sqrt{\frac{80 \times 920}{958{,}000} + \frac{83 \times 917}{849{,}000}} \text{ per } 1000\right.$$

$$\left. = 3 \pm 0{\cdot}39, \text{ which is significant.}\right]$$

(2) The average fatality rate of a certain disease is 12·4 per cent., but under a new treatment 180 people were treated with 15 deaths. Can one say that the treatment has reduced the fatality rate of that disease?

[*Answer*, Expected number of deaths $= \dfrac{180 \times 12{\cdot}4}{100} = 22.$

$\sigma = \sqrt{180 \times 0{\cdot}124 \times 0{\cdot}876} = 4{\cdot}42.$

Difference between expected and actual number of deaths $= 22 - 15 = 7 \pm 4{\cdot}42$, which is not significant.]

(3) In breeding certain stocks, 408 hairy and 126 glabrous plants were obtained. If the expectation is 25 per cent. glabrous, is the divergence significant, or might it have occurred as a fluctuation of sampling? (Udny Yule.)

$$\sigma = \sqrt{0{\cdot}25 \times 0{\cdot}75 \times 534} \text{ (see p. 400).}$$
$$= 10.$$

As the expectation is 25 per cent. $= \frac{1}{4} \times 534 = 133{\cdot}5$, and the observed result is 126,

∴ Difference between observed and theoretical result is

$$133{\cdot}5 - 126 = 7{\cdot}5.$$

Hence the observed difference falls well within the standard deviation of the expected number and might therefore occur as the result of fluctuations of sampling.

Chauvenet's Criterion.—The arithmetical mean is, as we have seen, calculated by dividing the sum of the various observation data by the total number of observations—provided all the observations have been made with equal care and that their deviations from the true value or mean are purely fortuitous and not due to any extrinsic cause (such as error in arithmetical work, etc.). The question therefore arises, **How can we (in cases of doubt) tell whether any one or more of the errors or deviations from the mean probably are not fortuitous and are therefore to be rejected for statistical purposes?** The simplest test is the one worked out by Chauvenet, and is therefore called *Chauvenet's Criterion* for suspecting the reliability of observations whose deviations from the mean appear to be too great to be merely fortuitous.

Let x/σ be the magnitude of any error or deviation expressed in terms of σ. Then the probability of the occurrence of any error outside the limits $\pm x/\sigma$ is expressed by $1 - A$ (where A = area of curve within the ordinates at the abscissal points $\pm x/\sigma$, see fig. 152 and p. 412). Hence, if N = total number of observations, $N(1 - A)$ represents the number of observations whose deviations from the mean are numerically greater than $\pm x/\sigma$.

Therefore if $N(1 - A)$ is less than 1/2, the *probability* of such deviations occurring fortuitously is less than the probability of the occurrence of observations whose deviations are numerically less than $\pm x/\sigma$. In other words, there is a greater probability of errors outside $\pm x/\sigma$ *not* occurring fortuitously than there is of their occurring in that way.

Hence when $N(1 - A) = \frac{1}{2}$, *i.e.* $\dfrac{2N - 1}{2N} = A$,

the observation errors lying outside XN and X_1N_1 probably are not fortuitous.

But when the value of A or $A/2 (= a)$ is known, the value of x/σ is known at once from the probability integral table. Hence a knowledge of N and of σ gives one a criterion for the assessment of the reliability of any particular observation.

It is to be noted, however, that no measurement may be rejected as worthless on the strength of Chauvenet's Criterion alone if the observer is satisfied that it was made with due care. It is just such observations that often lead to great discoveries (*e.g.* the discovery of argon and its congeners as the result of the observed greater density of atmospheric nitrogen than that of nitrogen prepared from ammonia).

EXAMPLES.

(1) In a series of 250 observations on the duration of pregnancy, the standard deviation from the mean (281·68) has been found to be 15 days. The observations gave values ranging from 190 to 330 days. Find what values may be excluded as improbably fortuitous (*e.g.* due to pathological causes, errors in dating the beginning of pregnancy, etc.).

Applying Chauvenet's Criterion we have

$$A = \frac{2 \times 250 - 1}{2 \times 250} = 0{\cdot}998, \ \textit{i.e.}\ \frac{A}{2} = 0{\cdot}499.$$

$\therefore$ $\dfrac{x}{\sigma}$ (as given in the probability integral table) $= 3{\cdot}09$.

But $\sigma = 15$, $\therefore$ $x = 3{\cdot}09 \times 15$, *i.e.* $\pm 46{\cdot}4$.

Hence any observation which gives a deviation from 281·68 days greater than $\pm 46{\cdot}4$ days is not likely to occur fortuitously.

Therefore, a first application of Chauvenet's Criterion makes the duration of pregnancy as not less than 235(= 281·68 − 46·4) and not more than 328(= 281·68 + 46·4) days.

[*Note.*—By applying Chauvenet's Criterion to the observations remaining after excluding those whose duration is less than 235 days and more than 328 days, it will be found that the probable duration of pregnancy ranged between 242 and 321 days. A third application of the Criterion narrows down the limits to 243 to 320 days. After this, no further rejections are allowable.]

(2) A. J. Clark found the following to be the rates of the rabbit's excised heart at various temperatures. Find whether these figures are in agreement with the Van't Hoff-Arrhenius formula, viz.,

$$\frac{K_2}{K_1} = e^{\frac{Q}{2}\frac{(T_2-T_1)}{T_1T_2}} \quad \text{(p. 281).}$$

15° C.	25	30° C.	82	38° C.	170.
25° C.	64	34° C.	120		

Applying the formula we obtain $\frac{K_{25}}{K_{15}} = \frac{64}{25} = 2{\cdot}56 = e^{\frac{Q}{2}\cdot\frac{10}{298\times288}}$, giving $Q = 16130$. Similarly $\frac{K_{30}}{K_{15}}$ gives $Q = 13810$; $\frac{K_{34}}{K_{15}}$ gives $Q = 14600$; and $\frac{K_{38}}{K_{15}}$ gives $Q = 14930$. Therefore mean $Q = 14867$.

$$\therefore \quad \text{Standard deviation} = \sqrt{\frac{(14867-16130)^2+(14867-13810)^2+\ .\ .\ .}{4}}$$

i.e. $\sigma = 835.$

Applying Chauvenet's Criterion, we get maximum deviation probable is that value of x/σ which makes

$$A = \frac{2\times4-1}{2\times4} = 0{\cdot}875,$$

or

$$\frac{A}{2} = 0{\cdot}4375, \text{ viz. } \frac{x}{\sigma} = 1{\cdot}334.$$

$$\therefore \quad x = 1{\cdot}334\times835 = 1114.$$

As the maximum deviation in the given cases is only 1263, it is quite a probable one. Hence the figures are in agreement with the Van't Hoff-Arrhenius law.

The following are the observed and calculated results:—

Observed	$K_{25} = 64{\cdot}0$	$K_{30} = 82{\cdot}0$	$K_{34} = 120{\cdot}0$	$K_{38} = 170{\cdot}0$
Calculated	$K_{25} = 59{\cdot}5$	$K_{30} = 89{\cdot}6$	$K_{34} = 123{\cdot}4$	$K_{38} = 174{\cdot}3$

(W. M. Feldman and A. J. Clark, *The Lancet*, 1924, vol. i.)

EXERCISE.

In a series of twenty-eight observations in a certain psychological experiment, the mean was found to be 55 and $\sigma = 6{\cdot}98$. Find the limit (x_1) of allowable deviation from the mean.

[Here $A = \frac{2\times28-1}{2\times28} = 0{\cdot}98214. \quad \therefore \quad \frac{A}{2} = 0{\cdot}4911.$

$\therefore \quad x/\sigma = 2{\cdot}37. \qquad \therefore \quad x = 6{\cdot}98\times2{\cdot}37 = 16{\cdot}5.$

$\therefore$ All observations greater than 72 and less than 38 are not altogether reliable.]

Theorem of Least Squares.—If a number of observations are made upon a quantity and the errors of each of these noted, *i.e.* as nearly as can be estimated, then from a knowledge of these errors it is possible to find the most probable value of the quantity **on the assumption that the errors follow the normal distribution.**

Let n observations be taken, and let the errors be $x_1, x_2, \ldots x_n$.

Also suppose all measurements to be equally good, *i.e.* the precision of reading, measured by σ (see p. 400), being the same throughout.

The probability of an error lying between x_1 and $x_1+\delta x$ will be

$$\mathrm{P}_1 = \delta x \times y_1$$

where

$$y_1 = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x_1^2}{2\sigma^2}}$$

i.e.

$$\mathrm{P}_1 = \delta x \times \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x_1^2}{2\sigma^2}}$$

Similarly,

$$\mathrm{P}_2 = \delta x \times \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x_2^2}{2\sigma^2}}$$

Now, since x_1 and x_2 are quite independent,

$\therefore$ the probability of the two errors falling within the range δx from x_1 and x_2 respectively will be the product of the separate probabilities (Law II.),

i.e.

$$\mathrm{P}_1 \times \mathrm{P}_2,$$

or

$$\frac{\delta x \,.\, e^{-\frac{x_1^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \times \frac{\delta x \,.\, e^{-\frac{x_2^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$= \frac{(\delta x)^2}{2\sigma^2\pi}e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)}$$

Similarly, the probability of n errors $x_1, x_2, \ldots x_n$ falling within the range δx from $x_1, x_2, x_3, \ldots x_n$ respectively is given by the equation

$$\mathrm{P} = \frac{(\delta x)^n}{\sigma^n(2\pi)^{n/2}} \,.\, e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2+\ \ldots\ +x_n^2)}$$

$$= \mathrm{K}e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2+\ \ldots\ +x_n^2)}$$

$$= \frac{\mathrm{K}}{e^{\frac{1}{2\sigma^2}\Sigma x^2}}\left(\text{where K stands for } \frac{(\delta x)^n}{\sigma^n(2\pi)^{n/2}}\right)$$

$\therefore$ P will be a maximum when $e^{\frac{1}{2\sigma^2}\Sigma x^2}$ is a minimum, *i.e.* when Σx^2 is a minimum. In other words, *the most probable value of the quantity to be determined is that which makes the sum of the squares of the errors least.*

Application of Method of Least Squares.—(1) Let the observed magnitude y depend upon x in such a way that

$$y = mx + b,$$

representing a straight line.

It is required to find the most probable values of m and b.—For perfect accuracy we must have for the varying observations

$$\begin{aligned} y_1 &= mx_1 + b, \text{ so that } & y_1 - mx_1 - b &= 0, \\ y_2 &= mx_2 + b, \quad ,, & y_2 - mx_2 - b &= 0, \\ &\ldots & &\ldots \\ y_n &= mx_n + b, \quad ,, & y_n - mx_n - b &= 0. \end{aligned}$$

Let the actual observation equations found be

$$\left.\begin{aligned} y_1 - mx_1 - b &= A_1 \\ y_2 - mx_2 - b &= A_2 \\ \ldots & \\ y_n - mx_n - b &= A_n \end{aligned}\right\} \text{ where } A_1, A_2, \ldots A_n \text{ are the actual deviations.}$$

By theorem of least squares we ought to have that

$$A_1^2 + A_2^2 + \ldots + A_n^2 \text{ is a minimum,}$$

i.e. $$\Sigma(y - mx - b)^2 \text{ is a minimum,}$$

$$\therefore \quad \frac{\delta}{\delta b}[\Sigma(y - mx - b)^2] = 0,$$

i.e. $$\Sigma(y - mx - b) = 0 \quad . \quad . \quad \text{(i)}$$

and $$\frac{\delta}{\delta m}[\Sigma(y - mx - b)^2] = 0,$$

i.e. $$\Sigma x(y - mx - b) = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

If there are n observations, then

(i) becomes $\Sigma y - m\Sigma x - nb = 0$, or $m\Sigma x + bn = \Sigma y$. (1)

and (ii) becomes $\Sigma(xy) - m\Sigma(x^2) - b\Sigma x = 0$, or $m\Sigma(x^2) + b\Sigma x = \Sigma(xy)$ . . . . (2)

These can be solved for m and b in the usual manner.

Thus, multiplying (1) by Σx and (2) by n, we get

$$m(\Sigma x)^2 + bn\Sigma x = \Sigma y\Sigma x \quad . \quad . \quad (1\ a)$$
$$mn\Sigma(x^2) + bn\Sigma x = n\Sigma(xy) \quad . \quad . \quad (2\ a)$$

By subtraction, $m[(\Sigma x)^2 - n\Sigma(x^2)] = \Sigma y\Sigma x - n\Sigma(xy)$,

whence $$m = \frac{\Sigma y\Sigma x - n\Sigma(xy)}{(\Sigma x)^2 - n\Sigma(x^2)}.$$

Similarly, multiplication of (1) by $\Sigma(x^2)$, and (2) by Σx gives

$$b = \frac{\Sigma(xy)\Sigma x - \Sigma y \,.\, \Sigma(x^2)}{(\Sigma x)^2 - n\Sigma(x^2)}$$

Corollary.—From this result it follows that if

$$\Sigma x = 0 \quad \text{and} \quad \Sigma y = 0$$

then $$m = \frac{-n\Sigma(xy)}{-n\Sigma(x^2)} = \frac{\Sigma(xy)}{\Sigma(x^2)}$$

and $$b = 0$$

These are two important results in statistical theory, because they teach us that if we take the means of the observations $x_1, x_2, x_3, \ldots x_n$ and $y_1, y_2, y_3, \ldots y_n$ as the origin, that is, when $b = 0$ (see p. 113), then

$$m = \frac{\Sigma(xy)}{\Sigma(x^2)}$$

EXAMPLE.

Find the most probable values of m and b in the equation

$$s = mt + b$$

representing the solubility of $NaNO_3$, using the data given in the example on p. 353.

t	s	t^2	st
-6	68·4	36	-410·4
0	72·9	0	0
20	87·5	400	1750
40	102	1600	4080
$\Sigma t = 54$	$\Sigma s = 330{\cdot}8$	$\Sigma t^2 = 2036$	$\Sigma(st) = 5419{\cdot}6$

$$\therefore \quad m = \frac{\Sigma t \,.\, \Sigma s - n\Sigma(st)}{(\Sigma t)^2 - n\Sigma(t^2)} = \frac{54 \times 330{\cdot}8 - 4 \times 5419{\cdot}6}{54^2 - 4 \times 2036}$$

$$= \frac{3815{\cdot}2}{5228} = 0{\cdot}73.$$

$$b = \frac{\Sigma t \,.\, \Sigma(st) - \Sigma t^2 \,.\, \Sigma s}{(\Sigma t)^2 - n\Sigma(t^2)} = \frac{54 \times 5419{\cdot}6 - 2036 \times 330{\cdot}8}{54^2 - 4 \times 2036}$$

$$= \frac{380850{\cdot}4}{5228} = 72{\cdot}9.$$

$\therefore$ The equation is $s = 0{\cdot}73t + 72{\cdot}9.$

(2) If the equation is a parabola of the form $y = a + bx + cx^2$, then the numerical values of the constants a, b, c which will

best fit the n sets of data obtained in the laboratory are found by solving the three equations obtained by the partial differentiation of the expression $\Sigma(y-a-bx-cx^2)$, with respect to a, b and c, viz.

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad . \quad . \quad . \quad \text{(i)}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad . \quad . \quad \text{(ii)}$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad . \quad . \quad \text{(iii)}$$

solving in the usual manner for a, b and c.

If the equation is a parabola of the nth order, viz.

$$y = a + bx + cx^2 + dx^3 + \; . \; . \; . \; + mx^n,$$

then we must derive $(n+1)$ equations of similar types to (i), (ii) and (iii) (by partially differentiating with respect to a, b, c, d, . . . m).

Similarly in the case of any other function, such as a logarithmic function,

$$y = a + bx + c \log x,$$

the values of a, b and c are obtained by solving the derived equations

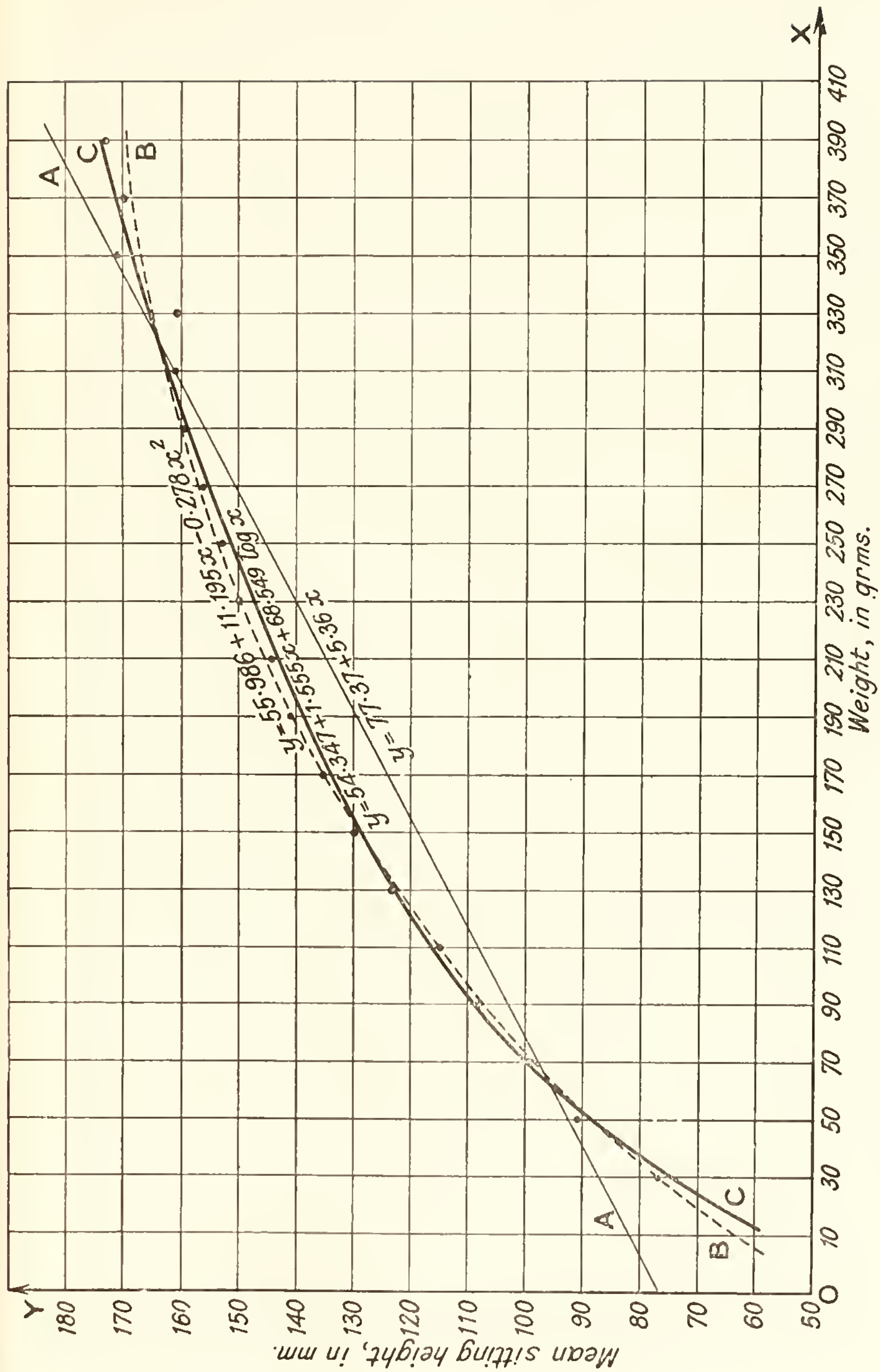
$$\Sigma y = na + b\Sigma x + c\Sigma \log x, \quad . \quad . \quad . \quad (1)$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x \log x \quad . \quad . \quad . \quad (2)$$
$$\Sigma y \log x = a\Sigma \log x + b\Sigma x \log x + c\Sigma (\log x)^2 \quad . \quad (3)$$

EXAMPLE.

(Modified from Pearl's "*Medical Biometry*.")

The following table gives a few of the pairs of measurements of embryos (weights x in grammes, sitting heights y in millimetres) out of a total of 20 such pairs indicated by the 20 black dots in fig. 156:—

Weight Interval of Embryo in Grammes.	Mean Sitting Height in mm. (y).	Midweight of Embryo in 20-gramme Unit Intervals, starting from 10 grammes (x).	xy.	x^2y.	$y \log x$.
0–19	58·8	1	58·8	58·8	0
20–39	76·4	2	152·8	305·6	22·9964
40–59	91·1	3	273·3	819.9	43·4638
...	...	...	...	...	...
...	...	...	...	...	...
340–359	171·0	18	3,078·0	55,404·0	214·6563
360–379	169·5	19	3,220·5	61,189·5	216·7566
380–399	173·6	20	3,472·0	69,440·0	225·8536
Totals	2,673·7 (Σy)	210 $= (\Sigma x)$	31,610·5 $= (\Sigma xy)$	453,700·3 $= (\Sigma x^2y)$	2,678·6433 $= (\Sigma y \log x)$



Mean sitting height, in mm.
Weight, in grms.
Y
X
O
A
B
C
$y = 55{\cdot}986 + 11{\cdot}195x - 0{\cdot}278x^2$
$y = 54{\cdot}347 + 1{\cdot}555x + 68{\cdot}549 \log x$
$y = 77{\cdot}37 + 5{\cdot}36x$

FIG. 156.

Find the equation of the graph that will fit these data best.

(*a*) Try first a straight-line graph.

The derived equations for a straight line are (p. 424):—

$$\Sigma y = m\Sigma x + nb,$$
$$\Sigma(xy) = m\Sigma(x^2) + b\Sigma x.$$

Putting $\Sigma y = 2673\cdot7$, $\Sigma x = 210$, $n = 20$ and $\Sigma(xy) = 31{,}640\cdot5$,

also $\Sigma x^2 = 1^2 + 2^2 + 3^2 + \ . \ . \ . + 20^2 = \frac{20 \times 21 \times 41}{6}$ (p. 71)

$= 2870,$

we obtain $2673\cdot7 = 210m + 20b,$

and $31{,}640\cdot5 = 2870m + 210b,$

whence $b = 77\cdot37$ and $m = 5\cdot36.$

$\therefore$ The equation of the best fitting line for the data is

$$y = 5\cdot36x + 77\cdot37.$$

As will be seen from fig. 156 this straight line (AA) does not fit the data in question at all well.

(*b*) We therefore try a parabola of the second order of the form

$$y = a + bx + cx^2$$

The derived equations are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2, \text{ or } 2673\cdot7 = 20a + 210b + 2870c,$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3, \text{ or } 31{,}640\cdot5 = 210a + 2870b + 44{,}100c,$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4, \text{ or } 453{,}700\cdot3 = 2870a + 44{,}100b + 722{,}666c.$$

Whence $a = 55\cdot986$, $b = 11\cdot195$ and $c = -0\cdot278$.

$\therefore$ Equation for best fitting parabola, the dotted curve BB, is

$$y = 55\cdot986 + 11\cdot195x - 0\cdot278x^2.$$

From the figure it is seen that although the parabola fits the data much better, the fit is not a very good one.

(*c*) We therefore use a logarithmic equation. The simplest one to use would be $y = c \log x$, but a formula like this would make $y = 0$ when $x = 1$. As in our case $y = 58\cdot8$ when $x = 1$, we must introduce a term a which raises the level of the curve (a being the y intercept (see p. 113)). Further, as for higher values of x the curve tends to straighten out more than a purely logarithmic curve would do, its equation must also contain a straight line term bx. We therefore take as our equation

$$y = a + bx + c \log x.$$

The derived equations (see p. 426) are

$$\underbrace{\Sigma y}_{2{,}673\cdot7} = \underbrace{na}_{20a} + \underbrace{b\Sigma x}_{210b} + \underbrace{c\Sigma (\log x)}_{18\cdot3861246c} \quad \left(\sum_{x=1}^{x=20} (\log x) \text{ being } 18\cdot3861246\right)$$

$$\underbrace{\Sigma(xy)}_{31{,}640\cdot5} = \underbrace{a\Sigma x}_{210a} + \underbrace{b\Sigma(x^2)}_{2{,}870b} + \underbrace{c\Sigma(x \log x)}_{230\cdot0033043c} \quad \left(\sum_{x=1}^{x=20} (x^2) \text{ being } 2870 \text{ (p. 71)}\right.$$

$$\left.\text{and } \sum_{x=1}^{x=20} x \log x \text{ being } 230\cdot0033043\right)$$

$$\underbrace{\Sigma(y \log x)}_{2678\cdot6433} = \underbrace{a\Sigma (\log x)}_{18\cdot3861246a} + \underbrace{b\Sigma(x \log x)}_{230\cdot0033043b} + \underbrace{c\Sigma (\log x)^2}_{19\cdot2694686c}$$

$$\left(\sum_{x=1}^{x=20} (\log x)^2 \text{ being } 19\cdot2694686\right).$$

Solving these simultaneous equations we obtain as the equation for the curve

$$y = 54{\cdot}347 + 1{\cdot}555x + 68{\cdot}549 \log x.$$

As seen from the figure, the fit of this curve with the data is very satisfactory.

Theorem.—The arithmetical mean of a series of observed values is the most probable value of the quantity measured.

Proof.—Let $x_1, x_2, x_3, x_4, \ldots x_n$, be the respective observations (n in number). Let their arithmetic mean $= \bar{x}$,

so that $$\bar{x} = \frac{\Sigma x}{n}$$

Let x_p = the most probable value of the magnitude.

Then $(x_1 - x_p), (x_2 - x_p), (x_3 - x_p), \ldots$ = residual errors.

Now the probability of making this system of errors is by Law II., p. 389, given by

$$P = \frac{1}{\sigma\sqrt{2\pi}} e^{-h^2\{\Sigma(x_1 - x_p)^2\}}, \quad \text{where} \quad h = \frac{1}{\sigma\sqrt{2}}$$

But, if x_p is the most probable value, it is obvious that the probability of making the series of errors $(x_1 - x_p), (x_2 - x_p)$, etc., must be a maximum.

Hence $\Sigma(x_1 - x_p)^2$ must be a minimum.

$$\therefore \quad \Sigma x_1 - \Sigma x_p = 0 \text{ (p. 424)},$$

or $$\Sigma x_p = \Sigma x_1.$$

But $$\Sigma x_p = n x_p.$$

$$\therefore \quad x_p = \frac{\Sigma x_1}{n} = \bar{x} \text{ (the arithmetical mean).}$$

EXAMPLE.

The following four observations of a certain quantity were made with equal care:—

10, 9·4, 10·3, 10·1.

Find the most probable value.

$$\text{The A.M.} = \frac{10 + 9{\cdot}4 + 10{\cdot}3 + 10{\cdot}1}{4} = \frac{39{\cdot}8}{4} = 9{\cdot}95.$$

The most probable value is therefore 9·95. This can be verified by actual calculation.

The residual errors are

$$(10 - 9{\cdot}95), \quad (9{\cdot}4 - 9{\cdot}95), \quad (10{\cdot}3 - 9{\cdot}95), \quad (10{\cdot}1 - 9{\cdot}95),$$

i.e. $$0{\cdot}05, \quad -0{\cdot}55, \quad 0{\cdot}35, \quad 0{\cdot}15.$$

∴ Sum of squares of residual errors

$$= (0{\cdot}05)^2 + (0{\cdot}55)^2 + (0{\cdot}35)^2 + (0{\cdot}15)^2 = 0{\cdot}45.$$

Now let us assume the most probable value to be less than 9·95, *e.g.* 9·6. Then sum of squares of residual errors

$$= (0{\cdot}4)^2 + (-0{\cdot}2)^2 + (0{\cdot}7)^2 + (0{\cdot}5)^2 = 0{\cdot}94.$$

Similarly, if the most probable value be taken as greater than 9·95, *e.g.* 10·2, then the sum of the squares of the residual errors

$$= (-0{\cdot}2)^2 + (-0{\cdot}8)^2 + (0{\cdot}1)^2 + (0{\cdot}1)^2 = 0{\cdot}7.$$

But both 0·94 and 0·7 are greater than 0·45. Similarly for any other figure higher or lower than 9·95. Hence the arithmetical mean is the most probable value.

Unsymmetrical Distribution.—Let us now consider a case where the distribution is not symmetrical, *e.g.* instead of the white and black balls (p. 388) being equally distributed, let the bag contain, say, one white ball and five black ones. The probability p of drawing a white is 1/6 and that (q) of drawing a black is 5/6 ($p+q = 1$).

Now, since by Law II. the probability of two independent events occurring together is the product of their separate probabilities, therefore the possibilities and the respective probabilities for any number of draws 1, 2, 3, . . . n are those shown in the following table:—

Number of Draws.	Different Possibilities.	Different Probabilities.
1	W, B	p, q
2	W^2, $2WB$, B^2	p^2, $2pq$, q^2
3	W^3, $3W^2B$, $3WB^2$, B^3	p^3, $3p^2q$, $3pq^2$, q^3
. . .	. . . .	. . . .
n	W^n, $nW^{n-1}B$, $\frac{n(n-1)}{1.2}W^{n-2}B^2$, . . . B^n	p^n, $np^{n-1}q$, $\frac{n(n-1)}{1.2}p^{n-2}q^2$, . . q^n

In other words, the different possibilities and probabilities are respectively given by the successive terms of the expansions $(W+B)^n$ and $(p+q)^n$.

We can now construct a frequency table, showing the probabilities of getting 0, 1, 2, 3, . . . r, . . . n whites (or other similar successes) in n draws (or other similar events), the probability in each case being denoted by the proportional frequencies of these different successes.

(1) Number of Successes. (x).	(2) Frequency. (f).	(3) Product of Numbers in Columns (1) and (2). (fx).	(4) Product of Numbers in Columns (1) and (3). (fx^2).
0	q^n	0	0
1	$nq^{n-1}p$	$nq^{n-1}p$	$nq^{n-1}p$
2	$\frac{n(n-1)}{1.2}q^{n-2}p^2$	$n(n-1)q^{n-2}p^2$	$2n(n-1)q^{n-2}p^2$
. . .	. . . .	. . . .	. . . .
r	$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}q^{n-r}p^r$	$\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots(r-1)}q^{n-r}p^r$	$\frac{rn(n-1)\ldots(n-r+1)}{1.2.3\ldots(r-1)}q^{n-r}p^r$
. . .	. . . .	. . . .	. . . .
n	p^n	np^n	n^2p^n
Sum	$\Sigma f = (q+p)^n = 1$	$\Sigma(fx) = np$	$\Sigma(fx^2) = np[1+(n-1)p]$

The curve would not be symmetrical, the median, mean and mode not being coincident (see p. 398 and fig. 157).

The sum of the frequencies (see column 2), *i.e.* Σf, is obviously

$$q^n + nq^{n-1}p + n\frac{(n-1)}{1\,.\,2}q^{n-2}p^2 + \ \ldots \ + p^n = (q+p)^n = 1$$

[since $(p+q) = 1$].

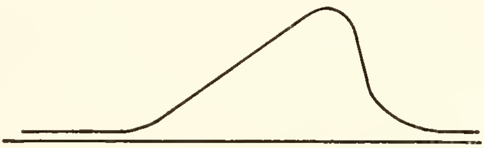


FIG. 157.—Unsymmetrical Distribution.

The sum of the terms in column 3 is

$$\Sigma(fx) = np[q^{n-1} + (n-1)q^{n-2}p + \ \ldots \ + p^{n-1}]$$
$$= np(q+p)^{n-1} = np = 1st\ moment\ about\ the\ origin,$$

taking the origin at the first term of the binominal.

The sum of the terms in column 4 is

$$\Sigma(fx^2) = nq^{n-1}p + 2n(n-1)q^{n-2}p^2 + \frac{3n(n-1)(n-2)}{1\,.\,2}q^{n-3}p^3 + \ \ldots \ + n^2p^n$$

$$= np\left[\left\{q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{1\,.\,2}q^{n-3}p^2 + \ \ldots \ + p^{n-1}\right\} + \left\{(n-1)q^{n-2}p + \frac{2(n-1)(n-2)}{1\,.\,2}q^{n-3}p^2 + \ \ldots \ + (n-1)p^{n-1}\right\}\right]$$

$$= np[(q+p)^{n-1} + (n-1)p(q+p)^{n-2}]$$
$$= np[1 + (n-1)p] = 2nd\ moment\ about\ the\ origin.$$

The *arithmetic mean* of the distribution is equal to the sum of the terms in column 3 divided by the sum of the terms in column 2,

$$= \frac{\Sigma(fx)}{\Sigma(f)} = \frac{np}{1} = np.$$

The *mean square deviation referred to zero* or "no success" as origin

$$= \frac{\Sigma(fx^2)}{\Sigma(f)} = np[1 + (n-1)p], \text{ since } \Sigma f = 1.$$

Hence the *standard deviation* σ (see p. 399), which is *the root mean square deviation from the arithmetic mean np*, is given by

$$\begin{aligned}\sigma^2 &= np[1+(n-1)p]-n^2p^2 \text{ (see p. 405).}\\ &= np(1-p)\\ &= npq \text{ (since } 1-p = q);\\ \therefore\quad \sigma &= \sqrt{npq}.\end{aligned}$$

$$\left(\text{If}\qquad p = q,\quad \sigma = \frac{\sqrt{n}}{2},\quad \text{or}\quad \sigma^2 = \frac{n}{4}.\right)$$

This formula teaches us two important facts:—

(1) When p is the same, σ varies as the square root of n, so that the larger the size of the sample, the larger is the value of σ. Thus, if p is the fatality rate of a certain illness like pneumonia, and $1-p$ is the recovery rate, then for 100 people suffering from that disease $\sigma = \sqrt{100p(1-p)} = 10\sqrt{p(1-p)}$, while for 400 people suffering from the same disease $\sigma = \sqrt{400p(1-p)} = 20\sqrt{p(1-p)}$, *i.e.* twice as large. The importance of this has already been pointed out (p. 417).

(2) Since $\sigma = \sqrt{np(1-p)}$ it follows that when p is very small $1-p$ is sensibly the same as 1 and $\sigma = \sqrt{np}$.

Skewness.—Since a frequency curve can be evolved from the binomial expansion $(p+q)^n$ it must vary in shape according as p and q are equal or unequal if n is not fairly large. If $p = q$, then we get the normal frequency curve in which there is a perfect balance between the frequencies of observation on either side of the maximum ordinate. In such a curve, therefore, the mean, median and mode coincide. If p and q are unequal, but neither is very small or very large, say when neither is less than about 0·1 or greater than about 0·9, the normal curve will still give a fair fit, provided n is large. If, however, p and q are unequal, and either p or q is very small, then the curve is unsymmetrical about its maximum ordinate. This asymmetry or lack of symmetry of a curve is called its *skewness* and is indicated by the fact that the mean, median and mode do not coincide. If the mean is greater than the mode then the curve is negatively skew; if less it is positively skew. Thus the frequency curve (Price-Jones curve) of the sizes of the red cells in a case of pernicious anæmia is negatively skew (shifted to the right). In a case of acholuric jaundice it is positively skew (shifted to the left). Skewness is also indicated when the sum of the positive deviations from the median is not equal numerically to the sum of the negative deviations. The degree of skewness can therefore be measured by means of either of these facts.

Measurement of Skewness.—There are several methods by which skewness can be measured.

Pearson has defined skewness as

$$\frac{(\text{mean} - \text{mode})}{\sigma}$$

$$\text{or} \quad 3\,\frac{(\text{mean} - \text{median})}{\sigma}$$

where σ = standard deviation.

A more exact measure is

$$\text{skewness} = \frac{\sqrt{\beta_1}(\beta_2+3)}{2(5\beta_2 - 6\beta_1 - 9)} \quad \text{(see p. 444).}$$

Equation of an Asymmetrical Probability or Frequency Curve. —The equation $y = Ae^{-\frac{x^2}{2\sigma^2}}$ applies to the normal probability curve resulting from the binomial expansion of $(p+q)^n$ in which $p = q = \frac{1}{2}$ and $n = \infty$. When, however, p is not equal to q (but $p+q = 1$), then we have

$$(p+q)^n = p^n + \frac{np^{n-1}q}{1} + \frac{n(n-1)}{1\,.\,2}p^{n-2}q^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}p^{n-3}q^3 + \;.\;.\;.\;.$$

Proceeding in an exactly similar manner to that in the case of the normal curve, we arrive at the differential equation

$$\frac{dy}{dx} = \frac{2y[2(n+1)qc - (p+q)(2x-c)]}{c[2(n+1)qc + (p-q)(2x-c)]} \quad . \quad . \quad . \quad (A)$$

which becomes identical with the equation on p. 407, viz.

$$\frac{dy}{dx} = \frac{2y}{(n+1)c}\left(n+2-\frac{2x}{c}\right) \text{ if we make } p = q = \tfrac{1}{2},$$

and we are led to the equation for the normal probability curve.

When, however, p is not equal to q, equation (A) becomes of the type $\dfrac{dy}{dx} = \dfrac{y(a-x)}{(\beta+\gamma x)}$, where a, β and γ are terms involving the constants p, q and c.

By transferring the origin to the point $(a, 0)$ the equation becomes

$$\frac{dy}{dx} = -\frac{yx}{\beta+\gamma(x+a)} = -\frac{yx}{\delta+\gamma x} \text{ (where } \delta = \beta+\gamma a).$$

$$\therefore \quad \int\frac{dy}{y} = -\int\frac{xdx}{\gamma x+\delta} = -\frac{1}{\gamma}\int\frac{\gamma x}{\gamma x+\delta}dx = -\frac{1}{\gamma}\int\frac{[(\gamma x+\delta)-\delta]}{\gamma x+\delta}dx$$

$$= -\frac{1}{\gamma}\int dx + \frac{1}{\gamma}\int\frac{\delta}{\gamma x+\delta}dx$$

$$\therefore \quad \log_e y = -\frac{x}{\gamma} + \frac{\delta}{\gamma^2}\log_e(\gamma x + \delta) + \log_e \mathrm{A} \text{ (where } \log_e \mathrm{A} = \text{integration constant).}$$

$$\therefore \quad y = \mathrm{A}e^{-\frac{x}{\gamma}}(\gamma x + \delta)^{\frac{\delta}{\gamma^2}} = y_0 e^{-kx}\left(1 + \frac{x}{a}\right)^{ka} \left(\text{where } k = \frac{1}{\gamma}, \text{ and } a = \frac{\delta}{\gamma}\right),$$

since A is the value of y when $x = 0$.

From the manner of its derivation, the type of curve represented by this equation is called a **skew binomial probability curve,** as contrasted with the **symmetrical binomial** or normal frequency curve.

Pearson's Generalised Probability Curve.—In practice, the two types of curves—symmetrical and skew binomial—do not cover all the distributions that commonly arise.

Now, as the only essential difference between the differential equations from which these curves are derived is in respect of the denominator, which in the case of the skew binomial contains a term in x and in the case of the symmetrical binomial contains no such term (*i.e.* the coefficient b of x is zero) it is reasonable to expect that a differential equation—not, of course, derivable from $(p+q)^{\infty}$, and therefore in a sense empirical—in which the denominator is a function of x, *i.e.* of the form $c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$, would lead on integration to the most general type of probability curve, which would cover all possible statistics. Pearson, however, found that an equation of the form $\frac{dy}{dx} = \frac{y(x+d)}{ax^2+bx+c}$, in which the denominator contains no higher power of x than x^2, leads on integration to a set of curves (12 in number) depending on the relative values of a, b and c, which represent adequately most of the statistics commonly met with.

Without going too deeply into the subject, we shall indicate the nature of the seven best known types of curves resulting from this equation, for different values of a, b and c, always assuming a to be positive.

Since $ax^2 + bx + c = a\left(x + \frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b-\sqrt{b^2-4ac}}{2a}\right)$ (see p. 20) different values of a, b and c will give different types of curves as follows:—

Type I.—*When c is negative.* $\sqrt{b^2-4ac}$ is positive and greater than b.

Therefore $\frac{b+\sqrt{b^2-4ac}}{2a}$ is positive (equal to, say, α_1), and $\frac{b-\sqrt{b^2-4ac}}{2a}$ is negative (equal to, say, $-\beta_1$).

$$\therefore \quad ax^2 + bx + c = a(x+\alpha_1)(x-\beta_1)$$

Therefore the differential equation becomes

$$\frac{dy}{dx} = \frac{y(x+d)}{a(x+\alpha_1)(x-\beta_1)}$$

which, on transferring the origin to the point $(-d, 0)$, becomes

$$\frac{yx}{a(x-d+\alpha_1)(x-d-\beta_1)}$$

$$= \frac{yx}{a(x+\alpha)(x-\beta)}$$

where $\alpha = (\alpha_1 - d)$ and $\beta = (\beta_1 + d)$

$$= \frac{y}{a(\alpha+\beta)}\left(\frac{\alpha}{(\alpha+x)} - \frac{\beta}{(\beta-x)}\right) \text{ (see p. 30).}$$

$$\therefore \int\frac{dy}{y}, \textit{ i.e. } \log_e y = \frac{\alpha}{a(\alpha+\beta)}\log_e(\alpha+x) + \frac{\beta}{a(\alpha+\beta)}\log_e(\beta-x) + \log_e C,$$

or

$$y = C(\alpha+x)^{k\alpha}(\beta-x)^{k\beta} \quad \left(\text{where } k = \frac{1}{a(\alpha+\beta)}\right)$$

$$= y_0\left(1+\frac{x}{\alpha}\right)^{k\alpha}\left(1-\frac{x}{\beta}\right)^{k\beta} \quad (\text{where } y_0 = C\alpha^{k\alpha}\beta^{k\beta})$$

Type II.—When $b = 0$. The denominator of the differential equation becomes ax^2+c, which equals $a\left(x+\sqrt{-\frac{c}{a}}\right)\left(x-\sqrt{-\frac{c}{a}}\right)$, which is of the same form as $a(x+\alpha)(x-\alpha)$. The differential equation is thus similar to that which produces Type I., except that $\beta = \alpha$. Therefore the resulting curve will have an equation of the form

$$y = y_0\left(1+\frac{x}{\alpha}\right)^{k\alpha}\left(1-\frac{x}{\alpha}\right)^{k\alpha} = y_0\left(1-\frac{x^2}{\alpha^2}\right)^{k\alpha}$$

Type III.—When $a = 0$ the differential equation becomes that of the *skew binomial* leading to the curve $y = y_0e^{-kx}\left(1+\frac{x}{\alpha}\right)^{k\alpha}$ (see p. 434).

Type IV.—When $b^2 < 4ac$, $\frac{\sqrt{b^2-4ac}}{2a}$ is imaginary, say, $l\sqrt{-1}$.

$$\therefore \quad \frac{dy}{dx} = \frac{y(x+d)}{a\left\{\left(x+\frac{b}{2a}\right)^2 + l^2\right\}}$$

which on transferring the origin to $\left(-\frac{b}{2a}, 0\right)$

$$= \frac{y\left\{x - \left(\frac{b}{2a} - d\right)\right\}}{a(x^2+l^2)}$$

$$\therefore \quad \log_e y = \frac{1}{2a}\log_e(x^2+l^2) - \frac{\left(\frac{b}{2a}-d\right)}{al}\tan^{-1}\frac{x}{l} + \log_e C \text{ (see pp. 243, 245)}$$

which is of the form $\quad y = y_0\left(1+\frac{x^2}{l^2}\right)^{-m} e^{-n\tan^{-1}x/l}$

Type V.—When $b^2 = 4ac$, we have

$$\frac{dy}{dx} = \frac{y(x+d)}{a\left(x+\frac{b}{2a}\right)^2}$$

which on transferring the origin to $\left(-\frac{b}{2a}, 0\right)$

$$= \frac{y\left\{x - \left(\frac{b}{2a} - d\right)\right\}}{ax^2}$$

$$\therefore \quad \log_e y = \frac{1}{a}\log_e x - \frac{\left(d - \frac{b}{2a}\right)}{ax} + \log_e C$$

which is of the form $\qquad y = y_0 x^{-m} e^{-\frac{n}{x}}$

Type VI.—When $b^2 > 4ac$, both roots are positive and the equation is

$$\frac{dy}{dx} = \frac{y(x+d)}{a(x+\alpha)(x+\beta)}$$

which, after resolving into partial fractions, transferring the origin to the point $(-\beta, 0)$ and integrating, yields the equation

$$y = y_0(x-\alpha)^m x^{-n}$$

Type VII.—When $a = b = 0$, *i.e.* when the denominator of the differential equation has no x terms. The equation is that of the *symmetrical binomial*, leading to the normal frequency curve

$$y = y_0 e^{-\frac{x^2}{2\sigma^2}}$$

There are several other types of probability curves, but for information regarding them the student must refer to Professor Karl Pearson's Memoir (*Phil. Trans.*, A, ccxvi, 1916, p. 429), or to W. P. Elderton's book "*Frequency Curves and Correlations.*"

It should be stated that frequency systems other than Pearson's have been proposed by Edgeworth and others, but none of them has succeeded in replacing Pearson's system.

Note.—In all the foregoing formulæ it will be seen that y_0 is the value of y when $x = 0$, and is therefore the height of the ordinate at the origin.

Pearson's Criterion.—The analysis briefly outlined in the foregoing shows that the type of curve is determined from the relation between b^2 and $4ac$, *i.e.* whether $b^2/4ac$ is negative (Type I.), zero (Type II.), <1 (Type IV.), $= 1$ (Type V.), >1 (Type VI.), or ∞ (Type III.). This ratio $b^2/4ac$, designated by the Greek letter κ, is therefore called the criterion of the distribution. When $a = 0$ and $b = 0$ the curve is the normal type (Type VII.). There are simple relationships between the various constants a, b, c and the moments of the distribution about the mean which enable one to determine the type of curve covering the given statistical data (see p. 444).

All those natural phenomena whose occurrence depends upon the multiplicity of uncontrollable causes which we call

chance are found to be capable of being represented by some sort of frequency curve. If the chance of the event happening is equal to that of its not happening, then the resulting curve is the *normal* or *symmetrical frequency curve.* Thus, in the case of heights of individuals, some of the numerous uncontrollable causes constituting the even chance that any individual will be shorter or taller by a specified number of inches than the mean of the general population are: heredity, race, social position, occupation, pose of body, physical health, nature of environment with respect to food, air, sleep, sunshine, etc. during the growth period, physical training, etc. If, however, the probability of an event happening is either greater or less than that of its not happening, then the resulting frequency curve assumes an *asymmetrical* shape. Thus, if instead of taking a *random* sample of men, we were to *select* a number of individuals coming from tall stock, so that the chance of tall individuals occurring in the sample was greater than that of the occurrence of short individuals, the frequency curve when plotted would assume an asymmetrical shape in such a way that there would be a higher frequency on the tall side of the mean than on the other side.

Although the factors responsible for the production of skew curves may be those of pure chance, such curves may also occur in cases where ordinarily the chance of the event happening is the same as that of its not happening but, as the result of some superimposed factor, the balance of chances is disturbed. When, therefore, one finds a skew curve where one would *a priori* expect a normal frequency curve, one should endeavour to ascertain the nature of the disturbing factor producing the skewness. Thus, if one finds the heights, weights or intelligence of the scholars in a large school plot out as a markedly skew curve it would probably be found that there is a preponderance of scholars belonging to a special class in respect to race, social status, etc. On the other hand, most of the statistics relating to the incidence of disease, mortality rates at different ages, degree of fertility, are as a rule markedly skew.

The following equations represent types of distribution of interest to the biomathematician:—

Type I.—The variation in the total rate of infant mortality in 1918 in American cities of over 25,000 population:

$$y = 46{\cdot}5502\left(1 + \frac{x}{2{\cdot}8946}\right)^{4{\cdot}4296}\left(1 - \frac{x}{14{\cdot}5155}\right)^{22{\cdot}2126}$$

(Raymond Pearl).

Type III.—The variation in the infectious disease rates in 1914 in 241 large English towns:

$$y = 55{\cdot}3e^{-0{\cdot}701x}\left(1 + \frac{x}{1{\cdot}87}\right)^{1{\cdot}31}$$

Type IV.—The variation in the rates of infant mortality in 1918 among whites in rural counties of the United States of America (Raymond Pearl):

$$y = 0{\cdot}005609\left[1 + \frac{x^2}{(4{\cdot}5095)^2}\right]^{-12{\cdot}5407} e^{25{\cdot}9862 \tan^{-1} x/4{\cdot}5095}$$

Type V.—The distribution of frequencies of specimens of Anemone nemdrosa with different numbers of sepals (Yule):

$$y = Ax^{-9{\cdot}64}e^{-17{\cdot}1/x}$$

Type VI.—The variation in the rate of infant mortality among the coloured in American rural counties in 1918:

$$y = 4{\cdot}2887 \times 10^{27}(x - 20{\cdot}9629)^{3{\cdot}0627}x^{-19{\cdot}9804}$$

Statistical Curve Fitting.—We have seen in Chapter XXII., p. 353, that having obtained a number of laboratory results in a particular investigation, we plot a graph from the corresponding pairs of values of the two variables and then endeavour not only to determine the class of curve (straight line, parabola, hyperbola, exponential curve, etc.) to which this particular graph belongs, but also by evaluating the constants in the general equation of the curve, establish the particular equation connecting the two variables, and thus determine the "law" of the phenomenon under investigation. When we have to deal with masses of statistical data we are confronted with similar tasks, viz.:

(1) We have to decide whether the particular distribution agrees with that of a normal frequency group, and, if not, which of the other types of frequency curves best agrees with the distribution under consideration.

(2) Having found the nature of the curve which best fits in with our data, we set out to determine the constants of the curve from the observed statistics.

The process of finding which particular class of frequency curves best agrees with the particular distribution and then of evaluating the constants so as to determine the equation of the particular curve, constitutes what is called *curve-fitting*.

The process of curve-fitting will be best understood by working out an actual numerical example.

The table below illustrates the distribution of marks obtained by 514 candidates in a certain examination.

Marks Obtained. (x)	Number of Candidates. (f)
1–5	5
6–10	9
11–15	28
16–20	49
21–25	58
26–30	82
31–35	87
36–40	79
41–45	50
46–50	37
51–55	21
56–60	6
61–65	3
Total . .	514

Find (A) what type of curve will best agree with this distribution, (B) what is the exact equation of the curve so found, (C) how far do the given frequency data correspond with the frequencies calculated from the equation, *i.e.* what is the "goodness of the fit"?

To find the *type of curve* suitable for any distribution, the simplest method is to assume the distribution to be normal, and on this assumption to calculate the theoretical frequencies of the various groups. The significance of any discrepancies between the calculated and observed frequencies is then estimated by special tests—such as the χ^2 or "chi-square" test (p. 443). If the discrepancies are not significant, then we say that the normal frequency curve covers the distribution. If, however, the discrepancies are significant, then it is necessary to evaluate (from a knowledge of the first, second, third and fourth moments of the distribution) the coefficients a, b and c in the denominator $ax^2 + bx + c$ of the general differential equation for a frequency curve (p. 434), and from the numerical relation between them establish the type of curve (see pp. 434–6). It is, of course, possible to fit the curve straight away from the numerical relation between the coefficients a, b and c, but the labour is very considerable, and as most distributions are normal, it is much simpler to assume the given distribution also to be normal until it is discovered not to be so.

We shall therefore assume the distribution of the marks to be normal and the curve $y = y_0 e^{-\frac{x^2}{2\sigma^2}}$ to apply.

We shall need to compute the mean and the standard deviation.

(i) *To Find the Mean of the Given Distribution.*—Since the variate x in the present case varies discretely or discontinuously, *i.e.* at definite and fixed intervals of five marks (which we may call the *unit*), and not continuously (*i.e.* by infinitesimal increments), we must take the mid-value of each interval to represent the average value of the particular interval (*e.g.* 3 in the "1 – 5" interval, 8 in the "6 – 10" interval, and so on), and treat each of the candidates of any particular group as though he received the number of marks corresponding to the mid-value of the class-interval of his group. Thus each of the 5 candidates of the class-interval "1 – 5" is considered as having received 3 marks; each of the 9 candidates with the class-interval of "6 – 10" is considered as having received 8 marks, and so on. This assumption is, of course, not quite correct, and certain adjustments will be necessary to eliminate the concomitant errors. With these we shall deal later (see p. 441). If now we take some arbitrary but convenient number, such as 33 *marks* (which is the mid-value of the class-interval "31 – 35," *i.e.* the interval lying midway between the two extreme class-intervals), as the *origin*, and the number of marks in each class-interval (*i.e.* 5) as the *unit*, we get the following table:—

(1) Mean No. of Marks.	(2) Deviation from 33.	(3) Frequency.	(4) First Moment.	(5) Second Moment.	(6) Third Moment.	(7) Fourth Moment.
	x (5 marks as unit)	f	(fx)	(fx^2)	(fx^3)	(fx^4)
3	−6	5	−30	180	−1080	6480
8	−5	9	−45	225	−1125	5625
13	−4	28	−112	448	−1792	7168
18	−3	19	−117	441	−1323	3969
23	−2	58	−116	232	−464	928
28	−1	82	−82	82	−82	82
33	0	87	0	0	0	0
38	+1	79	+79	+79	+79	+79
43	+2	50	100	200	400	800
48	+3	37	111	333	999	2997
53	+4	21	84	336	1344	5376
58	+5	6	30	150	750	3750
63	+6	3	18	108	648	3888
—	—	514	−110	2814	−1646	41142
		$N = \Sigma f$	$N_1^1 = \Sigma fx$	$N_2^1 = \Sigma fx^2$	$N_3^1 = \Sigma fx^3$	$N_4^1 = \Sigma fx^4$

The *arithmetic mean* of the distribution is (see p. 403):

$$33 + 5(-\tfrac{110}{514}) = 31{\cdot}93.$$

(ii) *To Find the Standard Deviation of the Given Distribution.*—From column 5 of the table on p. 440 we have $\dfrac{\Sigma fx^2}{\text{N}} = \dfrac{2814}{514}$ $= \sigma_0^2$, the second moment coefficient about the arbitrary mean as origin in terms of the group-interval, viz. 5 marks, as the unit. Then from $\sigma^2 = \sigma_0^2 - d^2$ (p. 405), we have $\sigma^2 = \frac{2814}{514} - (\frac{110}{514})^2 = 5{\cdot}429$ group-interval units.

As stated, however, the value of σ^2 must be corrected for the errors inherent on the erroneous assumption that the mid-value of each group-interval actually represents the average number of marks obtained by the candidates of that group. This is easily done by means of **Sheppard's adjustment,** which consists in subtracting 1/12 or 0·083 from σ^2. It is unnecessary to show here how this figure is obtained, for although the requisite mathematics is not difficult, it is rather complicated and tedious. Taking this figure for granted we have

Corrected $\sigma^2 = 5{\cdot}429 - 0{\cdot}083 = 5{\cdot}346$ group-interval units.

$\therefore$ True $\sigma = \sqrt{5{\cdot}346} = 2{\cdot}312$ group-interval units.

$= 5 \times 2{\cdot}312 = 11{\cdot}56$ marks.

The Equation for the Normal Frequency Curve in this case is, therefore,

$$y = \frac{\text{N}}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}} = \frac{514}{2{\cdot}312\sqrt{2\pi}}e^{-\frac{x^2}{2\times 5{\cdot}346}} = 88{\cdot}7e^{-\frac{x^2}{10{\cdot}7}}$$

or if we take σ in units of marks instead of units of "5 marks" we have

$$y = \frac{514}{11{\cdot}56\sqrt{2\pi}}e^{-\frac{x^2}{2(11{\cdot}56)^2}} = 17{\cdot}8e^{-\frac{x^2}{2(11{\cdot}56)^2}}$$

Comparison of Calculated and Observed Group Frequencies.—This is most expeditiously done by arranging the various steps in tabular form (as shown in the table, p. 442) as follows:—

(*a*) Place in column 1 the upper limits of each group-interval. Thus, since 1 means 0·5 to 1·5, and 5 means 4·5 to 5·5 (see p. 4), therefore the upper limit of the interval 1 to 5, is 5·5. Similarly that of 6 to 10 is 10·5, and so on.

(*b*) Express the deviation (x) of each of these upper limits from the mean (31·93) as shown in column 2, in terms of σ as shown in column 3.

(*c*) The values of $(1 - \text{A})/2$ for each of the negative values of x/σ and of $(1 + \text{A})/2$ for each of the positive values of x/σ are recorded in column 4. These give the frequencies of all the

groups between 0·5 mark and any deviation x_n/σ (when the total frequency covered by the whole curve is taken as 1).

(*d*) The difference between each of the values in column 4 and the next below it, gives the frequency of the particular group comprised between x_n/σ and x_{n+1}/σ (in terms of total frequency as 1). This multiplied by N (*i.e.* 514 in our case) gives the calculated or theoretical frequency of the particular group. Thus for group 1 to 5, $(1-\mathrm{A})/2 = 0{\cdot}0111$, therefore the calculated frequency of that group is $514 \times 0{\cdot}0111 = 5{\cdot}7$. For groups 1 to 10 together $(1-\mathrm{A})/2 = 0{\cdot}0319$, therefore frequency of group 6 to 10 alone is $514(0{\cdot}0319 - 0{\cdot}0111) = 0{\cdot}0208 \times 514 = 10{\cdot}7$; and so on. Or, if we call the value of $(1-\mathrm{A})/2$ at any point x_n/σ, $(1-\mathrm{A}_n)/2$, and at the next point x_{n+1}/σ, $(1-\mathrm{A}_{n+1})/2$, then the calculated frequency of the group between x_n/σ and x_{n+1}/σ is $\dfrac{1-\mathrm{A}_{n+1}}{2} - \dfrac{1-\mathrm{A}_n}{2} = \dfrac{\mathrm{A}_n - \mathrm{A}_{n+1}}{2}$. These various values when multiplied by N give the frequencies of the various groups, as recorded in column 5 (see Example (1) on p. 414).

(1) Upper Limit of Group.	(2) Distance (x) of Upper Limit from Mean (31·93).	(3) $\frac{x}{\sigma} = \frac{x}{11{\cdot}56}$	(4) Value of $\frac{1 \mp \mathrm{A}}{2}$ at point $\frac{x}{\sigma}$	(5) Calculated Frequency (f_c) of Group $\frac{\mathrm{N}(\mathrm{A}_n - \mathrm{A}_{n+1})}{2}$	(6) Observed Frequency (f_o)	(7) $\frac{(f_c - f_o)^2}{(f_c)}$	Ordinate $y = \frac{\mathrm{N}}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ at point $\frac{x}{\sigma}$ (x here being the deviation of the Group Mean, *e.g.* 3, 8, etc. from 31·93).
	−ve	−ve	$\frac{1-\mathrm{A}}{2}$				
5·5	−26·43	−2·286	0·0111	0·0111 × 514 = 5·7	5	$\frac{(0{\cdot}7)^2}{5{\cdot}7} = 0{\cdot}09$	3·9
10·5	−21·43	−1·854	0·0319	0·0208 × 514 = 10·7	9	$\frac{(1{\cdot}7)^2}{10{\cdot}7} = 0{\cdot}27$	10·4
15·5	−16·43	−1·421	0·0776	etc. = 23·5	28	etc. = 0·86	23·2
20·5	−11·43	−0·988	0·1616	,, 43·2	49	,, 0·78	42·9
25·5	− 6·43	−0·556	0·2891	,, 65·5	58	,, 0·86	65·8
30·5	− 1·43	−0·124	0·4508	,, 83·1	82	,, 0·01	83·7
	+ve	+ve	$\frac{1+\mathrm{A}}{2}$				
35·5	+ 3·57	+0·309	0·6213	,, 87·6	87	,, 0·00	88·3
40·5	+ 8·57	+0·741	0·7706	,, 76·7	79	,, 0·07	77·3
45·5	+13·57	+1·174	0·8798	,, 56·1	50	,, 0·66	56·1
50·5	+18·57	+1·606	0·9159	,, 34·0	37	,, 0·26	33·7
55·5	+23·57	+2·039	0·9789	,, 17·0	21	,, 0·94	16·8
60·5	+28·57	+2·471	0·9932	,, 7·1	6	,, 0·26	7·0
65·5	+33·57	+2·901	0·9982	,, 2·6	3	,, 0·06	2·4
Totals				513·1	514	$\frac{\Sigma(f_c - f_o)^2}{f_c} = \chi^2 = 5{\cdot}12$	511·5

The "Chi-square" (χ^2) *Test for Goodness of Fit.*—It is obvious that since $(f_c - f_o)^2/f_c$ is a measure of the difference between the calculated and observed frequencies of each group, therefore χ^2 which is $\Sigma(f_c - f_o)^2/f_c$ (as given at the bottom of column 7) measures the difference between the total calculated and the total observed frequencies.

Thus when $\chi^2 = 0$ the agreement is perfect between the two sets of frequencies. On the other hand, when χ^2 is large the probability of the fit being a good one must be very slight. In other words, the goodness of the fit depends upon the magnitude of χ^2; the greater the value, the worse is the fit.

It also clearly depends upon the number of groups, the greater the number the better the fit (for the same value of χ^2). Tables have been constructed which give the probability (P) that any distribution consisting of n groups in which χ^2 has any particular value is a true example of the type of curve applied to it. From such a table we learn that when $\chi^2 = 5{\cdot}12$ and the number (n) of frequency groups is 13, then $P = 0{\cdot}948$ (about), which means that the chances are 948 to 52, *i.e.* 18 to 1, that the differences between the calculated and observed frequencies are no greater than would be expected from random sampling. Hence we can confidently say that the fit is a good one and the frequencies under consideration follow a normal distribution.

The table on p. 440 also gives in columns 6 and 7 the third and fourth moments of the distribution, from which the values of the coefficients a, b and c in the denominator of the differential equation of the generalised frequency curve (p. 434) can be calculated if it is found that a normal frequency curve does not fit the distribution. Thus, the third moment coefficient about the *arbitrary* mean $= -\frac{1646}{514} = -3{\cdot}2023$. Therefore, as can be shown, its value about the *true* mean is

$$-3{\cdot}2023 - 3 \times 5{\cdot}4289(-\tfrac{110}{514}) - (-\tfrac{110}{514})^3 = 0{\cdot}29296.$$

Also, the fourth moment coefficient about the *arbitrary* mean $= \frac{41142}{514} = 80{\cdot}0428$. This, again, can be shown to have a value about the *true* mean of

$$80{\cdot}0428 - 4 \times 0{\cdot}29296(-\tfrac{110}{514}) - 6 \times 5{\cdot}4289(-\tfrac{110}{514})^2 + (-\tfrac{110}{514})^4 = 78{\cdot}7964.$$

The third like the first moment coefficient needs no further correction, but the fourth like the second moment coefficient must be corrected for the assumption that the group mean represents the average number of marks received by the candidates in that group. Sheppard's adjustment in the case of the fourth moment coefficient consists in subtracting $(\frac{1}{2} \times 5{\cdot}4289 - 0{\cdot}02917)$, giving a true value of 76·11112.

If we call the first moment coefficient $\mu_1(= -\frac{110}{514})$
the adjusted second moment coefficient $\mu_2(= 5{\cdot}34558)$
,, ,, third ,, ,, $\mu_3(= 0{\cdot}29296)$
,, ,, fourth ,, ,, $\mu_4(= 76{\cdot}11112)$,

and if, also, we call the ratio $\dfrac{\mu_3{}^2}{\mu_2{}^3}$ β_1 ($= 0{\cdot}00056$ in our case)

and $$\frac{\mu_4}{\mu_2^2} = \beta_2 \; (= 2{\cdot}66365 \text{ in our case})$$

then it can be shown that the coefficients a, b and c can be expressed as follows:—

$$a = \frac{2\beta_2 - 3\beta_1 - 6}{2(5\beta_2 - 6\beta_1 - 9)} \; (= -0{\cdot}08 \text{ in our case}),$$

from which it follows that when β_1 *is approximately* 0 *and* β_2 *is approximately* 3, $a = 0$ (i.e. *the denominator of the equation on p. 434 contains no* x^2 *term*).

$$b = \frac{\sqrt{\mu_2\beta_1(\beta_2+3)^2}}{2(5\beta_2 - 6\beta_1 - 9)} \; (= -0{\cdot}03 \text{ in our case}),$$

from which it follows that when β_1 *is approximately* 0, $b = 0$ (i.e. *the denominator of the same equation contains no* x *term*).

In other words, *if in any distribution* β_1 *is found to be very small and* β_2 *is approximately equal to* 3, *then a normal curve may be fitted to the distribution.* This is so in the case we have been considering.

Further, $$c = \frac{\mu_2(4\beta_2 - 3\beta_1)}{2(5\beta_2 - 6\beta_1 - 9)},$$

so that κ, which equals

$$\frac{b^2}{4ac} \; (= -0{\cdot}0004 \text{ in our case})$$

being negligibly small, again indicates a normal distribution (see p. 436).

EXAMPLE.

3404 boys between fourteen and fifteen years old were weighed, and it was found that the following were the frequencies for the various weight intervals:—

Weight.	Number of Boys.	Weight.	Number of Boys.
65– 70	3	125–130	131
70– 75	9	130–135	76
75– 80	142	135–140	52
80– 85	301	140–145	20
85– 90	289	145–150	29
90– 95	380	150–155	14
95–100	416	155–160	10
100–105	404	160–165	2
105–110	315	165–170	2
110–115	320	170–175	5
115–120	262	175–180	1
120–125	221		

Calculate the mean weight and standard deviation and ascertain whether a normal curve will fit the given distribution.

Taking 102·5 as the origin, then

1st moment coefficient about origin $= \frac{874}{3404} = 0{\cdot}2568$.

$\therefore$ A.M. $= 102{\cdot}5 + 0{\cdot}2568 \times 5 = 103{\cdot}784$ lb.

2nd moment coefficient about origin $= \frac{37492}{3404} = 11{\cdot}014$.

$\therefore$ 2nd moment coefficient about mean $= 11{\cdot}014 - (0{\cdot}2568)^2 = 10{\cdot}948$.

Applying Sheppard's adjustment, we have

$$\sigma^2 = 10{\cdot}948 - \tfrac{1}{12} = 10{\cdot}865.$$

$\therefore \quad \sigma = \sqrt{10{\cdot}865} = 3{\cdot}296$, *i.e.* $3{\cdot}296 \times 5$ (5 being the unit) $= 16{\cdot}48$ lb.

3rd moment coefficient about origin $= \frac{107110}{3404} = 31 \cdot 466$.

$\therefore$ 3rd moment coefficient about mean

$= 31 \cdot 466 - 3 \times 10 \cdot 865 \times 0 \cdot 2568 - (0 \cdot 2568)^3 = 23 \cdot 08.$

4th moment coefficient about origin $= \frac{1502932}{3404} = 441 \cdot 519$.

$\therefore$ 4th moment coefficient about mean

$= 441 \cdot 519 - 4 \times 23 \cdot 08 \times 0 \cdot 2568 - 6 \times 10 \cdot 865 \times (0 \cdot 2568)^2 - (0 \cdot 2568)^4 = 413 \cdot 44.$

$\therefore$ Adjusted 4th moment coefficient $= 413 \cdot 44 - \frac{1}{2} \times 10 \cdot 865 + 0 \cdot 029$

$= 408 \cdot 04.$

$$\therefore \quad \beta_1 = \frac{(23 \cdot 08)^2}{(10 \cdot 865)^3} = 0 \cdot 416$$

$$\beta_2 = \frac{408 \cdot 04}{(10 \cdot 865)^2} = 3 \cdot 46.$$

As β_1 is not very small, a normal curve will not fit the distribution. Indeed, as

$$\kappa\left(= \frac{b^2}{4ac}\right) = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)} = -1 \cdot 04$$

which is negative, it follows that the type of curve which will best fit the distribution is Type I, viz.

$$y = y_0\left(1 + \frac{x}{a}\right)^{ka}\left(1 - \frac{x}{\beta}\right)^{k\beta}$$

The evaluation of a, β and k is beyond the scope of this book.

Correlation and Regression.

The reader is asked to consider the following cases:—

(1) The cubical expansion of a metal or gas with heat.

(2) The alteration in the density of a gas with change of temperature.

(3) The relation between the volume of a gas and the pressure to which it is subjected (at constant temperature).

(4) The relation between the height of a person and his weight.

(5) The relationship between the height of a person and his susceptibility to, say, heart disease.

(6) The relationship between a history of rheumatic fever and the presence of heart disease.

(7) The relation between overcrowding and infantile mortality.

(8) The relation between the vaccinal condition of a person and his susceptibility to small-pox.

In each of the first three cases there is within ordinary experimental limits a constant and perfectly definite relationship between the two measurements—although the nature of the relationship is different in each case, as we shall see presently—so that when the temperature of the metal or gas is increased or diminished by a fixed amount there is a concomitant perfectly fixed increase or diminution in the volume of the metal

or gas, so that we can tell the volume from the temperature, or the temperature from the volume; when the temperature of the gas is increased or diminished by a fixed amount there is a corresponding fixed diminution or increase in the density of the gas, and when the pressure of the gas is increased or diminished by a fixed amount there is a corresponding fixed diminution or increase in the volume of the gas. In each of these cases, therefore, in virtue of the perfect correspondence between the measurements, we can predict with absolute certainty the size of one variable when that of the other is known. **When there is such perfect correspondence between two sets of measurements, we say that there is complete or perfect correlation between them.**

Now take the relationship between the height of a person and his susceptibility to, say, heart disease. As far as we know there is absolutely no relationship at all between the two, so that we cannot in any way guess the condition of the heart from the person's height, and vice versa. In such a case, therefore, there is no concomitant variation between the two conditions, and we say that there is no correlation, or there is zero correlation, between height and the condition of the heart.

Again, let us take the relationship between height and weight, or the relationship between rheumatic fever and heart disease, or that between overcrowding and infantile mortality. We know from statistics and general experience that, although we cannot predict with certainty the weight of a person from his height, the condition of his heart from a past positive or negative history of rheumatic fever, or the extent of the infantile mortality in a given district from a knowledge of the density of the population in that district, yet there is a certain greater or lesser tendency to concomitant variation between the corresponding pairs of measurements in each of the three cases, so that **on the average** a tall person will be heavier than a short one, a person with a history of rheumatic fever will be more likely to have heart disease than one in whom there is no such history, and a district in which there is considerable overcrowding is likely to have a higher infantile mortality than one in which the density of the population is very low. In these cases, therefore, where there is only **a tendency to concomitant variation,** and there is no absolute correspondence between the pairs of measurements, we say there is **imperfect correlation** between them. And as the correspondence is such that an **increase** in one measurement leads us to expect an

increase in the other measurement, and vice versa, we say the correlation between them is of a **positive** kind.

Lastly, consider the relationship between vaccination and small-pox. It has, of course, been abundantly proved that vaccinated persons have a smaller susceptibility to small-pox than the non-vaccinated, so that communities with a *higher* percentage of vaccinated persons will, on the average, have a *smaller* small-pox mortality, and vice versa. Here, then, we have a case of imperfect correlation again, but of a **negative** kind; or we can say there is imperfect positive correlation between vaccination and immunity against small-pox. In the case of the density of a gas we can say that there is *perfect negative correlation* between the density and temperature.

Measurement of Correlation.—Perfect positive correlation, as in the case of the thermal expansion of metals, is indicated by $+1$; absence of correlation, as in the case of height and heart disease, is indicated by zero (0); whilst perfect negative correlation, as in the case of the density of a gas and temperature, or the relation between the pressure and volume of a gas, is indicated by -1. Hence we see that when the correlation is imperfect, as in the cases of the other examples which we considered in the preceding paragraphs, it must be represented by some decimal fraction ranging between 0 and $+1$ in the case of positive correlation and between -1 and 0 in the case of negative correlation. **This number lying between the limits ± 1, which measures the degree of correlation between two sets of measurements, is called a correlation coefficient** or **correlation ratio,** according as the nature of the graph representing the correlation is rectilinear (as in the ideal case of, say, thermal expansion, or any other example of simple interest law) or curvilinear, as in the ideal case of the pressure-volume law of a gas.

Method of Finding a Correlation Coefficient.—In the domain of physics it is perfectly easy to determine the correlation coefficient, because whether we plot y against x or x against y, the result will always be the same graph (except, of course, for errors of observation). Thus the line OA in fig. 158 gives not only the volume (y) of a gas at any temperature (x)—under constant pressure—but also the temperature of the gas for any volume—under the same conditions of pressure. Similarly, whether in the case of an ideal gas we plot the volume (y) of the gas against the pressure—at constant temperature—or the pressure (x) of the gas against the volume—under the same conditions of temperature—the result will be the same hyperbola (fig. 159). Hence we see that *when the two graphs*—

straight lines or curves—that result from plotting y against x and x against y coincide, the correlation is perfect, and the correlation coefficient or ratio is unity.

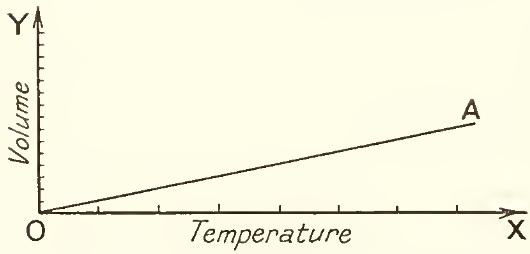


Fig. 158.—Perfect Rectilinear Correlation.

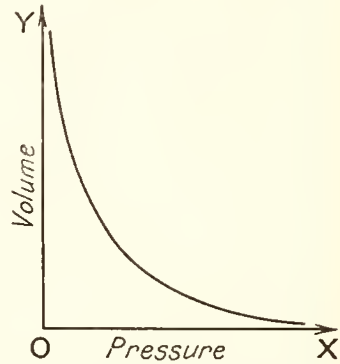


Fig. 159.—Perfect Curvilinear Correlation.

In cases of zero correlation, such as that between height and susceptibility to heart disease, if we plot the percentage number of people, y, suffering from a certain kind of heart disease for different heights, x, we get the line AB (fig. 160) parallel to the X axis, because whatever a person's height may be, his susceptibility to that particular type of heart disease is the same; if we plot the percentages of people of given height x having different kinds or different degrees of severity, y, of heart disease, we shall get the line CD parallel to the Y axis, because whatever the person's height, his susceptibility to the different

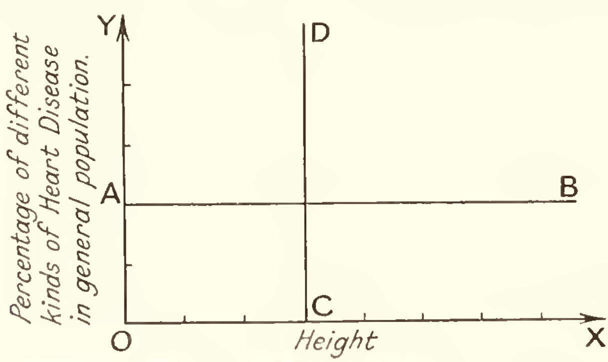


Fig. 160.—Zero Correlation.

types of heart disease is the same as that of the general population. *Zero correlationship is therefore indicated when the lines resulting from plotting each variable against the other are at right angles to each other.*

In cases of imperfect correlation, such as exists between, say, overcrowding and infant mortality, if we plot the average rates of mortality, y, corresponding to given densities of population, x, we shall get the line PQ, whilst if we plot the average

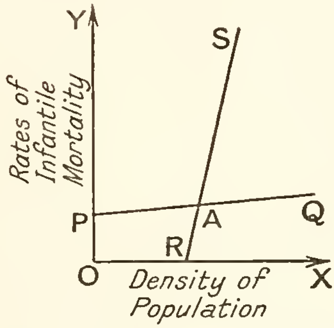


Fig. 161.—Imperfect Correlation. (Very low.)

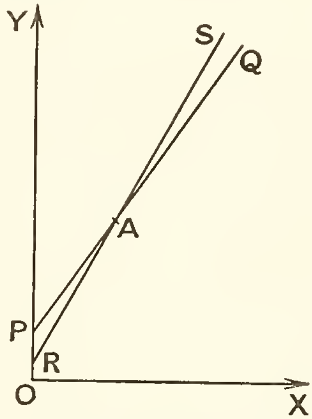


Fig. 163.—Imperfect Correlation. (Very high.)

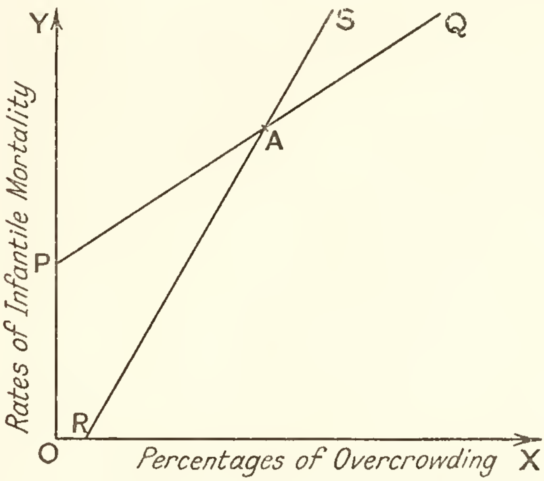


Fig. 162.—Imperfect Correlation. (Moderately high.)

densities of population corresponding to given rates of infantile mortality, we shall get the line RS cutting the line PQ at an acute angle. The size of the angle of intersection SAQ of such two lines varies inversely with the degree of correlationship between the two variables (see figs. 161–163).

Hence we see that *the angle SAQ is in some way a measure*

of the degree of correlationship between the two sets of measurements. It is this imperfect correlationship that one constantly meets with in biological and sociological phenomena.

EXAMPLE ON CORRELATION.

A close study of the following example will illustrate the method of finding the correlation coefficient between two sets of measurements.

The Census Report of 1911 gives the percentage of overcrowding (*i.e.* the percentage of the population which has less than one room per two persons) in each of twenty-nine metropolitan boroughs, together with the infantile mortality in each borough. It is required to find the correlation coefficient between overcrowding and infantile mortality.

The procedure is as follows:—

(1) Draw up a table of double entry showing the frequencies of given mortality intervals for the various percentage intervals of overcrowding (see table, p. 452). The *middle unbracketed figure* in each square denotes the *frequency* or number of boroughs having a percentage x_n of overcrowding, and y_n mortality.* Thus, four boroughs have 10 to 15 per cent. overcrowding and 120 to 130 infant mortality. (Each row or column in the table is called an *array*.)

(2) Treat each x interval as if its value were located at its mid-value (compare p. 440). Thus the x_1 interval, 0–5, is to be taken as 2·5, the x_2 interval, 5–10, is to be taken as 7·5, etc. The error involved in this assumption will be later corrected by means of Sheppard's adjustments (compare p. 441).

(3) Deal with the y intervals in the same way. Thus the interval y_1 (70–80) is to be taken as 75, the y_2 interval (80–90) is to be taken as 85, etc., and the necessary corrections will be made later.

(4) Find what mean value of y (infant mortality) is associated with any value of x (percentage of overcrowding). Thus the mean value of y associated with x = 5–10 (= 7·5) is

$$\frac{(75 \times 1 + 95 \times 1 + 105 \times 1 + 125 \times 1)}{4} = \frac{400}{4} = 100;$$

that associated with x = 10–15 (= 12·5) is

$$\frac{(105 \times 3 + 115 \times 1 + 125 \times 4 + 135 \times 1 + 145 \times 2)}{11} = 123{\cdot}18,$$

and so on. *Enter these values under the appropriate x columns as shown in the table.*

(5) Similarly, find what mean value of x is associated with any y. Thus the mean value of x (percentage of overcrowding) associated with infantile mortality y = 100–110 (= 105) is

$$\frac{2{\cdot}5 \times 1 + 7{\cdot}5 \times 1 + 12{\cdot}5 \times 3 + 22{\cdot}5 \times 1 + 27{\cdot}5 \times 1}{7} = \frac{97{\cdot}5}{7} = 13{\cdot}93,$$

and so on for the other y's. Enter the results at the ends of the appropriate rows as shown in the table (p. 452).

* For the meaning of the numbers enclosed in brackets above and below this number, see explanation on p. 453, as well as p. 455.

(6) Plot: (*a*) The various *actual* values of x and the corresponding *average* y's, viz. (2·5, 105), (7·5, 100), (12·5, 123·18), etc. These are shown by crosses in fig. 164, and PQ is the straight line which, as we shall see, fits these points best.

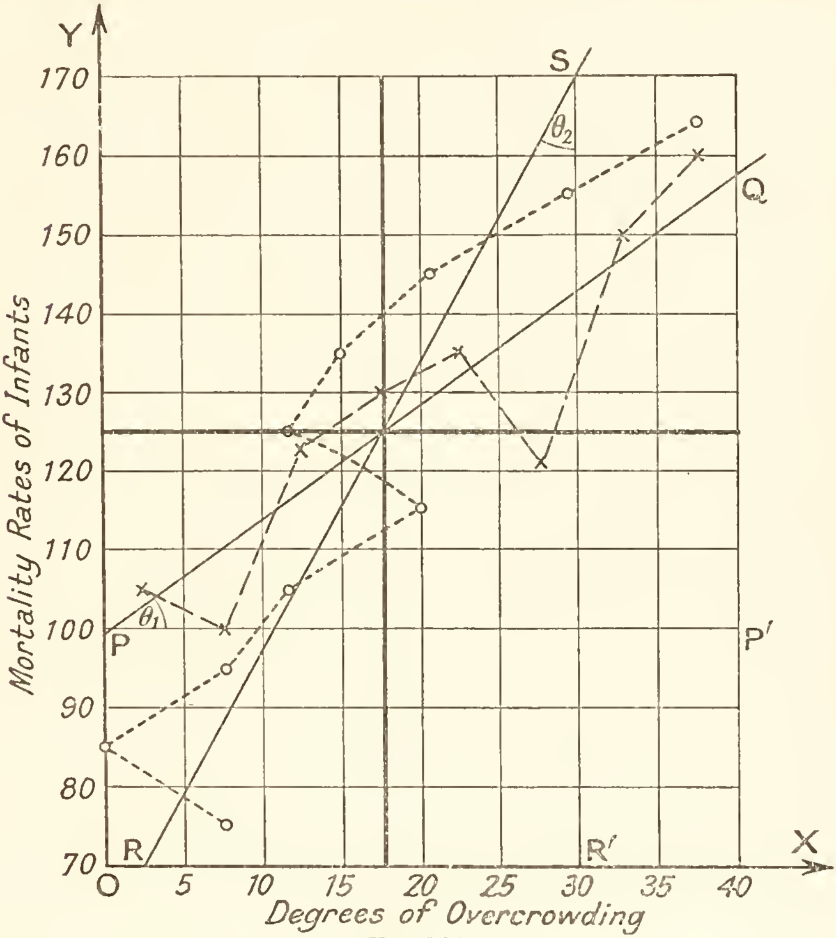


FIG. 164.

(*b*) The various *actual* values of y and the corresponding *average* x's, viz. (75, 7·5), (85, 0), (95, 7·5), (105, 13·93), etc. These are shown by small circles in fig. 164, and RS is the straight line which best fits these values.

These lines intersect at a point whose abscissa, x, can be proved to be the mean of all the x's, and whose ordinate, y, is the mean of all the y's, *i.e.* the point (17·85, 125·69).

(7) The mean of all the x's, as well as that of all the y's, is found as described on p. 402. Thus, taking 17·5 as the arbitrary mean of the x's, and 5 (the group-interval) as the unit, we obtain the table given on p. 453.

$$\text{Then A.M. of all the } x\text{'s} = 17{\cdot}5 + \frac{\Sigma f\xi}{\Sigma f} \times 5 \text{ (see p. 403).}$$

$$= 17{\cdot}5 + \tfrac{2}{29} \times 5$$

$$= 17{\cdot}85.$$

Similarly, taking 125 as the arbitrary mean of the y's, and 10 (the group-interval) as the unit, we have (see table, p. 453)

$$\text{A.M. of the } y\text{'s} = 125 + \tfrac{2}{29} \times 10 \quad = 125{\cdot}69.$$

Hence, *the point of intersection of the two lines PQ, RS is the point* (17·85, 125·69).

	Percentage of Overcrowding (x).									
Infant Mortality (y).	x_1 0–5 (−3).	x_2 5–10 (−2).	x_3 10–15 (−1).	x_4 15–20 (0).	x_5 20–25 (+1).	x_6 25–30 (+2).	x_7 30–35 (+3).	x_8 35–40 (+4).	Totals of y's.	Mean x for Individual Values of y.
y_1 70–80 (−5)	—	(10) 1 [10]	—	—	—	—	—	—	1	7·5
y_2 80–90 (−4)	—	—	—	—	—	—	—	—	0	0
y_3 90–100 (−3)	—	(6) 1 [6]	—	—	—	—	—	—	1	7·5
y_4 100–110 (−2)	(6) 1 [6]	(4) 1 [4]	(2) 3 [6]	—	(−2) 1 [−2]	(−4) 1 [−4]	—	—	7	13·93
y_5 110–120 (−1)	—	—	(1) 1 [1]	—	—	(−2) 1 [−2]	—	—	2	20
y_6 120–130 (0)	—	(0) 1 [0]	(0) 4 [0]	(0) 1 [0]	(0) 1 [0]	—	—	—	7	13·93
y_7 130–140 (+1)	—	—	(−1) 1 [−1]	(0) 1 [0]	—	—	—	—	2	15
y_8 140–150 (+2)	—	—	(−2) 2 [−4]	—	—	(4) 1 [4]	(6) 1 [6]	—	4	21·25
y_9 150–160 (+3)	—	—	—	—	(3) 2 [6]	—	(9) 1 [9]	(12) 1 [12]	4	28·75
y_{10} 160–170 (+4)	—	—	—	—	—	—	—	(16) 1 [16]	1	37·5
Total of x's	1	4	11	2	4	3	2	2	29 $= \Sigma f$	—
Mean y for Individual Values of x	105	100	123·18	130	135	121·67	150	160	—	—
$\Sigma_1 f\xi\eta$.	6	20	2	0	4	−2	15	28	73 $= \Sigma_2 f\xi\eta$	—

$$\Sigma f\xi = -3\times1-2\times4-1\times11+0+1\times4+2\times3+3\times2+2\times4 = 2$$
$$\Sigma f\eta = -5\times1-3\times1-2\times7-1\times2+1\times2+2\times4+3\times4+4\times1 = 2.$$

Note.—$\Sigma_2 f\xi\eta$ stands for the sum of all the $\Sigma_1 f\xi\eta$ terms.

The middle unbracketed figure in each square represents frequency f.

The figure above it enclosed in round brackets represents the product of ξ (the deviation of the particular x from the arbitrary mean of all the x's) by η (the deviation of the corresponding y from the arbitrary mean of all the y's), *i.e.* $\xi\eta$.

The figure below the frequency number enclosed in square brackets represents the product $f\xi\eta$.

(8) We shall want the *standard deviations of the x's and of the y's* (see p. 404). These can be easily evaluated from the tables.

The second moment coefficient about the origin (in the case of x)

$$= \frac{\Sigma f\xi^2}{\Sigma f} = \frac{102}{29} = 3{\cdot}517.$$

$\therefore$ Second moment coefficient about the mean

$$= 3{\cdot}517 - (\tfrac{2}{29})^2 = 3{\cdot}5170 - 0{\cdot}0048 = 3{\cdot}5122.$$

Percentage of Overcrowding. (Mid Value of the x Interval.)	Deviation from chosen Mean (17·5) (ξ).	Frequency (Number of Boroughs) (f).	($f\xi$).	($f\xi^2$).
2·5	−3	1	− 3	9
7·5	−2	4	− 8	16
12·5	−1	11	−11	11
17·5	0	2	0	0
22·5	+1	4	+4	4
27·5	+2	3	+6	12
32·5	+3	2	+6	18
37·5	+4	2	+8	32
Totals .		29 $= \Sigma f$	24 − 22 = 2 $= \Sigma f\xi$	102 $= \Sigma f\xi^2$

Infant Mortality. (Mid Value of the y Interval.)	Deviation from chosen Mean (125) η.	Frequency (f).	$f\eta$.	$f\eta^2$.
75	−5	1	− 5	25
85	−4	0	0	0
95	−3	1	− 3	9
105	−2	7	−14	28
115	−1	2	− 2	2
125	0	7	0	0
135	+1	2	+ 2	2
145	+2	4	+ 8	16
155	+3	4	+12	36
165	+4	1	+ 4	16
Totals .		29 $= \Sigma f$	26 − 24 = 2 $= \Sigma f\eta$	134 $= \Sigma f\eta^2$

$\therefore$ Adjusted second moment coefficient about the mean

$$= 3{\cdot}512 - \tfrac{1}{12} = 3{\cdot}512 - 0{\cdot}083 = 3{\cdot}429.$$

$\therefore$ *Standard deviation of* x is given by

$$\sigma_x = 5\sqrt{3{\cdot}429} = 5 \times 1{\cdot}85 = 9{\cdot}25$$

(5 being the group-interval unit).

Similarly, *standard deviation of* y is given by

$$\begin{aligned}\sigma_y = 10\sqrt{\tfrac{134}{29} - \left(\tfrac{2}{29}\right)^2 - \tfrac{1}{12}} &= 10\sqrt{4{\cdot}6207 - 0{\cdot}0048 - 0{\cdot}083} \\ &= 10\sqrt{4{\cdot}5329} \\ &= 21{\cdot}29.\end{aligned}$$

(9) We shall now find the equations of the lines PQ, RS.

Take the point of intersection (17·85, 125·69), which as we have seen represents the mean of all the x's and that of all the y's, as the origin.

Let equation of PQ be $y = mx$ (see p. 113). Then from p. 425 we know that when the point representing the means of the x's and of the y's is the origin, the best value of m is $\Sigma(xy)/\Sigma(x^2)$, where *x and y now represent the deviations of the different values of the respective variables from the means as origin* (*i.e.* are the same as ξ and η respectively in the tables on p. 453).

But $\dfrac{\Sigma(xy)}{n}$ = mean of all the products of the corresponding pairs of deviations $= p$ (say),

$$\therefore \quad \Sigma(xy) = np.$$

Also $$\frac{\Sigma(x^2)}{n} = \sigma_x^2 \text{ (where } \sigma_x = \text{S.D. of all } x\text{'s)},$$

$$\therefore \quad \Sigma(x^2) = n\sigma_x^2.$$

$$\therefore \quad m = \frac{np}{n\sigma_x^2} = \frac{p}{\sigma_x^2}$$

Hence *equation of the line $y = mx$ becomes*

$$y = \frac{p}{\sigma_x^2}x$$

which tells us that for each unit deviation of x from the mean of all the x's there is an average deviation of $\dfrac{p}{\sigma_x^2}$ in y from the mean of all the y's.

Similarly for the line RS we have

$$x = \frac{p}{\sigma_y^2}y$$

which tells us that for each unit deviation of y from the mean of all the y's there is an average deviation of p/σ_y^2 in x from the mean of all the x's. Hence each coefficient p/σ_x^2, p/σ_y^2, might reasonably be used as a measure of the correlation between x and y, except that σ_x^2 and σ_y^2 being as a rule different, the two coefficients are necessarily different. It is, however, possible to obtain a common correlation coefficient between x and y, or y and x, which will give us the amount of change (or deviation from the mean) of y corresponding to a unit change (or deviation from the mean) of x, and vice versa, when the changes or deviations are measured not in terms of their units of distribution (such as degree of overcrowding, or rate of infantile mortality), but in terms of their own units of variability, viz. x/σ_x and y/σ_y, respectively.

Thus the equation

$$y = \frac{p}{\sigma_x^2}x$$

is the same as

$$y = \frac{p}{\sigma_x}\frac{x}{\sigma_x}$$

Dividing throughout by σ_y we have

$$\frac{y}{\sigma_y} = \frac{p}{\sigma_x\sigma_y}\frac{x}{\sigma_x}$$

Similarly, the equation

$$x = \frac{p}{\sigma_y^2}y$$

is the same as

$$x = \frac{p}{\sigma_y}\frac{y}{\sigma_y}$$

which, on division throughout by σ_x, becomes

$$\frac{x}{\sigma_x} = \frac{p}{\sigma_x\sigma_y}\frac{y}{\sigma_y}$$

These equations tell us that for each unit deviation of x (or y) from its mean (in terms of its own standard deviation) there is an average deviation of y (or x) from its mean of $p/\sigma_x\sigma_y$ units in terms of its own standard deviation as unit.

The common coefficient $p/\sigma_x\sigma_y$ of x or y, which is generally designated by the letter r, is called the **Coefficient of Correlation** between x and y, or between y and x, and it may be defined as the quantity which measures, in terms of its respective standard deviation as unit, the average change in one variable corresponding to a unit change in the other variable.

In the problem under consideration, we have

$$r = \frac{p}{\sigma_x\sigma_y} = \frac{\Sigma xy}{29\sigma_x\sigma_y} = \frac{\Sigma xy}{29 \times 9{\cdot}25 \times 21{\cdot}29}$$

To evaluate r, it remains therefore to ascertain the numerical value of Σxy (where x and y are the respective deviations of the two variables from the true means as origin). To do this, we must first find the value of $\Sigma(\xi\eta)$, where ξ and η are the deviations of each x and y from the arbitrarily chosen means 17·5 and 125 as origins. This is easily done by entering just above the frequency number in each square the particular product $\xi\eta$, and just below the frequency number the particular product $f(\xi\eta)$, where f is the frequency number. *E.g.* row (100–110) is two class intervals less than the row containing the origin (120–130), and column (10–15) is one class interval less than the column containing the origin (15–20). Therefore the square common to row 100–110 and to column 10–15 has a product deviation $\xi\eta$ of $-2 \times (-1) = 2$ (shown in brackets above the frequency number 3) and a product $f\xi\eta$ of $3 \times 2 = 6$ (shown in square brackets below the frequency number 3), and so on for the others. Therefore $\Sigma_1 f(\xi\eta)$ is obtained by determining all the $f\xi\eta$ products in each column (or row) and then summing these results together. Thus,

$\Sigma_1 f\xi\eta$ in column $1 = 6$

,, ,, $2 = 10+6+4 = 20$,

and so on, as shown in the lowest row of the correlation table.

$$\therefore \quad \Sigma_2 f \xi \eta = \text{sum of all the numbers in the bottom row}$$
$$= 6+20+2+0+4-2+15+28$$
$$= 73.$$

$$\therefore \quad \Sigma xy \text{ (referred to the means as origin)}$$
$$= 73 - nd_x d_y = 73 - 29 \cdot \tfrac{2}{29} \cdot \tfrac{2}{29}$$
$$= 72{\cdot}862.$$

But since each x interval is 5 and each y interval is 10, therefore the value of each xy must be multiplied by 5×10, *i.e.* by 50.

$$\therefore \quad \Sigma xy = 72{\cdot}862 \times 50 = 3643{\cdot}1.$$

$$\therefore \quad r = \frac{3643{\cdot}1}{29 \times 9{\cdot}25 \times 21{\cdot}29} = 0{\cdot}64,$$

i.e. there is a very marked correlation between infant mortality and overcrowding.

The probable error of the correlation coefficient can be shown to be

$$0{\cdot}6745 \frac{1 - r^2}{\sqrt{n}}$$

provided $n > 25$. In our case, where $r = 0{\cdot}64$ and $n = 29$, the probable error is therefore

$$\frac{0{\cdot}6745(1 - 0{\cdot}64^2)}{\sqrt{29}} = \frac{0{\cdot}6745 \times 0{\cdot}5904}{5{\cdot}385}$$
$$= 0{\cdot}07,$$
$$\therefore \quad r = 0{\cdot}64 \pm 0{\cdot}07,$$

which means that a unit deviation from the mean of overcrowding (in terms of the S.D. of overcrowding) will produce, on the average, 0·64 of a unit deviation from the mean mortality rate (in terms of the S.D. of the mortality rate), with a probable error of $\pm 0{\cdot}07$; and vice versa.

Now that we know that $p/\sigma_x \sigma_y = r$, we can write the equations of the lines PQ and RS in the forms

$$\frac{y}{\sigma_y} = r \frac{x}{\sigma_x}, \quad \text{or} \quad y = \frac{r\sigma_y \cdot x}{\sigma_x},$$

and

$$\frac{x}{\sigma_x} = r \frac{y}{\sigma_y}, \quad \text{or} \quad x = \frac{r\sigma_x \cdot y}{\sigma_y}.$$

These two lines are called **Regression** or **Prediction Lines,** and the coefficients $r\sigma_y/\sigma_x$, $r\sigma_x/\sigma_y$ which measure the slopes of these lines are called **Regression Coefficients.** The lines are called prediction lines because one may use them for predicting the average value or change of one variable (y or x) associated with a known value or change of the other variable (x or y), see examples, p. 458. For the origin of the term regression, see p. 458.

Equations of Regression Lines.—Take again the equation

$$y = \frac{r\sigma_y \cdot x}{\sigma_x} = \frac{0{\cdot}64 \times 21{\cdot}29x}{9{\cdot}25} = 1{\cdot}47x.$$

Transferring the origin from (17·85, 125·69) to (0, 0), *the equation of the regression line of y on x* becomes

$$y - 125{\cdot}69 = 1{\cdot}47(x - 17{\cdot}85) \quad \text{or} \quad y = 1{\cdot}47x + 99{\cdot}45.$$

To draw the line put $x = 0$, when $y = 99{\cdot}45$. Therefore the line passes through (0, 99·45) and (17·85, 125·69). The line is PQ in fig. 164.

Similarly, from $x = \dfrac{r\sigma_x \cdot y}{\sigma_y}$, we have $x = 0{\cdot}277y$, which on transferring the origin to the point (0, 0) becomes

$$x = 0{\cdot}277y - 16{\cdot}97.$$

When $y = 70$, x becomes $= 2{\cdot}42$. Hence the line (RS in fig. 164) is determined by joining the points (2·42, 70) and (17·85, 125·69).

The **probable error** of a predicted mean value of x is

$$0{\cdot}6745\sigma_x\sqrt{1-r^2}.$$

Properties of Regression Lines.—From the equations

$$y = \frac{r\sigma_y \cdot x}{\sigma_x} \quad \text{and} \quad x = \frac{r\sigma_x \cdot y}{\sigma_y}$$

we learn that

(1) When $r = 0$ the two regression lines coincide with the x and y axes respectively (p. 448).

(2) When $r = \pm 1$ the two regression lines are coincident (p. 448).

(3) When $r = \pm 1$ and $\sigma_x = \sigma_y$ (*i.e.* the two characters are also equally variable), the regression lines not only coincide, but also bisect the angle between the axes.

(4) Since $\dfrac{r\sigma_y}{\sigma_x} = \tan\theta_1$ and $\dfrac{r\sigma_x}{\sigma_y} = \tan\theta_2$ (see fig. 164).

$$\therefore \quad r^2\left(= \frac{r\sigma_y}{\sigma_x} \times \frac{r\sigma_x}{\sigma_y}\right) = \tan\theta_1 \tan\theta_2.$$

Hence, when the complete distribution of the two variables is unknown, a very good idea of the size of the correlation coefficient between them may be obtained by drawing the regression lines if any two points on each of them (one of which may be the point of intersection) are known.

Thus, in fig. 164, $\tan\theta_1$ is seen to be 1·47 and $\tan\theta_2$ is seen to be 0·277.

$$\therefore \quad r^2 = 1{\cdot}47 \times 0{\cdot}277 = 0{\cdot}4, \text{ whence } r = \sqrt{0{\cdot}4} = 0{\cdot}63.$$

EXAMPLES ON REGRESSION.

(1) The mean height of 1000 fathers is 67·68 in., with S.D. = 2·70 in. The mean height of all the sons of these fathers is 68·65 in., with S.D. = 2·71 in. If r for stature between fathers and sons = 0·514, find the average height of sons whose fathers are 70 in. tall.

From equation $y = r\frac{\sigma_y}{\sigma_x}x$ we have

$$y = 0{\cdot}514 \times \frac{2{\cdot}71}{2{\cdot}70}x = 0{\cdot}516x.$$

But x in the case of fathers 70 in. high

$$= 70 - 67{\cdot}68 = 2{\cdot}32.$$

$\therefore \quad y = 0{\cdot}516 \times 2{\cdot}32 = 1{\cdot}197.$

$\therefore$ Height of sons $= 68{\cdot}65 + 1{\cdot}2 = 69{\cdot}85$ in.,

i.e. fathers whose height is 2·32 in. above the average have sons whose height is on the average only 1·2 in. above the general average.

Similarly, fathers whose height is 60 in., or 7·68 in. below the average height of all fathers, have sons whose height is on the average $0{\cdot}516 \times 7{\cdot}68$, or only 3·96 in. below the general average height of all sons (*i.e.* 68·65 − 3·96 = 64·69 in.). Hence we see that in the case of hereditary transmission of height there is a tendency on the part of the offspring to **go back** or **regress** from the condition of the fathers towards the general average of the population, so that children of very tall or very short parents are on the average neither so tall nor so short as their respective parents, but approximate more to the average height of the race. This is Galton's law of **filial regression,** and explains the origin of the term regression line, since the law of regression holds good for any pair of correlated variables.

(2) What should have been the expected infant mortality in Stepney in the year 1911 when the percentage of overcrowding was 35, and what should have been the expected degree of overcrowding in Hampstead where the infant mortality rate was 78?

$$y = 1{\cdot}47x + 99{\cdot}45 = 1{\cdot}47 \times 35 + 99{\cdot}45 = 150{\cdot}9 \text{ per } 1000.$$

The probable error of y is $\pm 0{\cdot}6745\sigma_y\sqrt{1-r^2} = \pm 0{\cdot}6745 \times 21{\cdot}29\sqrt{1-(0{\cdot}64)^2}$
$$= \pm 11{\cdot}1.$$

$\therefore$ Expected infant mortality in Stepney should have been between 139·8 and 162. (Actually it was 144.)

$$x = 0{\cdot}28y - 17 = 0{\cdot}28 \times 78 - 17 = 5 \text{ per cent.}$$

The probable error here is $\pm 0{\cdot}6745 \times 9{\cdot}25\sqrt{1-(0{\cdot}64)^2} = \pm 4{\cdot}7$.

$\therefore$ Expected percentage of overcrowding should have been between 0·3 and 9·7 per cent. (Actually it was 7·1 per cent.)

(3) The regression equation relating to heat production per hour and weight of infants in kilos is $H = 1{\cdot}341W + 1{\cdot}744$. What should be the heat production of an infant weighing 3 kilos?

$$H = 1{\cdot}341 \times 3 + 1{\cdot}744 = 5{\cdot}77 \text{ calories.}$$

EXERCISES.

(1) Two infants differ in weight by 100 grammes. What on the average should be the difference between their hourly heat productions?

[*Answer*, $1\cdot341 \times 0\cdot1 = 0\cdot134$ calories.]

(2) Two boroughs differed to the extent of 31 per cent. in respect of overcrowding in the year 1911. What should be the difference in their infant mortality rates in the same year?

[*Answer*, $1\cdot47 \times 31 = 45$ per 1000.]

The Standard Deviation of the Sum or Difference of Two Variables.—Let y and z be the respective means of the two variables and let δy and δz be the respective deviations from these means found in a particular sample. Then, if we write

$$u = y \pm z$$

we have
$$\delta u = \delta y \pm \delta z.$$

$$\therefore \quad (\delta u)^2 = (\delta y)^2 \pm 2\delta y \,.\, \delta z + (\delta z)^2$$

$\therefore$ If we have n samples, we shall obtain

$$\frac{\Sigma(\delta u)^2}{n} = \frac{\Sigma(\delta y)^2}{n} \pm \frac{2\Sigma(\delta y \delta z)}{n} + \frac{\Sigma(\delta z)^2}{n}$$

But $\dfrac{\Sigma(\delta u)^2}{n} = \sigma_u{}^2$, $\dfrac{\Sigma(\delta y)^2}{n} = \sigma_y{}^2$, and $\dfrac{\Sigma(\delta z)^2}{n} = \sigma_z{}^2$,

and since $r_{yz} = \dfrac{\Sigma(\delta y \delta z)}{n\sigma_y\sigma_z}$, therefore $\dfrac{2\Sigma(\delta y \delta z)}{n} = 2r\sigma_y\sigma_z$.

Hence $\sigma_u{}^2 = \sigma_y{}^2 \pm 2r\sigma_y\sigma_z + \sigma_z{}^2$.

From this it follows that, when y and z are absolutely independent of each other, so that $r = 0$, then

$$\sigma_u{}^2 = \sigma_y{}^2 + \sigma_z{}^2$$

$$\therefore \quad \sigma_u = \sqrt{\sigma_y{}^2 + \sigma_z{}^2}$$

i.e. the S.D. of a sum or difference of two uncorrelated means is equal to the square root of the sum of the squares of the S.D.'s of the two means.

Hence probable error of $u = 0\cdot6745\sqrt{\sigma_y{}^2 + \sigma_z{}^2}$ (see p. 418).

Skew Correlation.—When the regression lines are straight (of the form $y = a_1 + b_1x$, and $y = a_2 + b_2x$), then the degree of correlation between the variables x and y is measured by a **correlation coefficient**, r_{xy}, whose value, as we have seen, is $\sqrt{b_1b_2}$. When, however, the lines are curves (of the form $y = a_1 + b_1x + c_1x^2 + \ . \ . \ .$ and $y = a_2 + b_2x + c_2x^2 + \ . \ . \ .$), b_1 and b_2 no longer have the same meaning as in cases of rectilinear regressions (where they represent the slopes that the lines

make with the x and y axes respectively), and therefore the degree of relationship between x and y in such curvilinear or skew regression must be measured by another constant. The constant generally used is the **correlation ratio**, σ_{mx}/σ_x or σ_{my}/σ_y, where σ_{mx} or σ_{my} is the weighted standard deviation of the means of the arrays of x's or y's. *E.g.* in the case of overcrowding and infant mortality, σ_{mx} is the weighted S.D. of 7·5, 0, 7·5, 13·93, etc., *i.e.* of the means of the various degrees of overcrowding associated with corresponding rates of infant mortality (see table, p. 452). Its value is obtained from column 6 of Table A below, and is equal to $\sqrt{\frac{\Sigma(fd_x^2)}{n}}$.

σ_{my} is the S.D. of 105, 100, 123·18, etc., *i.e.* of the means of the various infant mortality rates associated with corresponding degrees of overcrowding (see table, p. 452). Its value is obtained from column 6 of Table B (p. 461) and is equal to $\sqrt{\frac{\Sigma(fd_y^2)}{n}}$.

σ_x and σ_y are the S.D.'s of the x's and y's in the distribution, whose values have already been found in the case under consideration (see pp. 453–4).

Table (A) for Calculating η_{xy}

(1) Infant Mortality Rates (y).	(2) Means of x Arrays. (Degrees of Overcrowding.)	(3) Deviation (d_x) from the Mean x of the Distribution (*i.e.* from 17·85).	(4) d_x^2	(5) Frequency of y (f)	(6) fd_x^2
75	7·50	− 10·35	107·1225	1	107·1225
85	0	− 17·85	318·6225	0	0
95	7·50	− 10·35	107·1225	1	107·1225
105	13·93	− 3·92	15·3664	7	107·5648
115	20·00	+ 2·15	4·6225	2	9·2450
125	13·93	− 3·92	15·3664	7	107·5648
135	15·00	− 2·85	8·1225	2	16·2450
145	21·25	+ 3·40	11·5600	4	46·2400
155	28·75	+ 10·90	118·8100	4	475·2400
165	37·50	+ 19·65	386·1225	1	386·1225
Totals				29 $= \Sigma f = n$	1362·467 $= \Sigma(fd_x^2)$

The correlation ratio $\eta = \sigma_{mx}/\sigma_x$ or σ_{my}/σ_y is therefore the ratio of the S.D.'s of the means of the arrays to the total S.D. of the distribution not arranged in arrays, and is a measure completely independent of the nature of the regression.

The computation of η between overcrowding and infant mortality is shown in Tables A and B.

From Table A, $\sigma_{mx} = \sqrt{\dfrac{\Sigma(fd_x^2)}{n}} = \sqrt{\dfrac{1362 \cdot 467}{29}} = 6 \cdot 85.$

$$\therefore \quad \eta_{xy} = \frac{\sigma_{mx}}{\sigma_x} = \frac{6 \cdot 85}{9 \cdot 25} = 0 \cdot 74.$$

From Table B, $\sigma_{my} = \sqrt{\dfrac{\Sigma(fd_y^2)}{n}} = \sqrt{\dfrac{7105 \cdot 924}{29}} = 15 \cdot 65.$

$$\therefore \quad \eta_{yx} = \frac{\sigma_{my}}{\sigma_y} = \frac{15 \cdot 65}{21 \cdot 29} = 0 \cdot 73$$

(substantially the same as η_{xy}, as one would expect).

Table (B) for Calculating η_{yx}

Degrees of Overcrowding (x).	Means of y Arrays. (Rates of Infant Mortality).	Deviation (d_y) from the Mean y of the Distribution (*i.e.* from 125·69).	d_y^2	Frequency of x (f)	fd_y^2
2·5	105·00	−20·69	428·0761	1	428·0761
7·5	100·00	−25·69	659·9761	4	2639·9044
12·5	123·18	− 2·51	6·3001	11	69·3011
17·5	130·00	+ 4·31	18·5761	2	37·1522
22·5	135·00	+ 9·31	86·6761	4	346·7044
27·5	121·67	− 4·02	16·1604	3	48·4812
32·5	150·00	+24·31	590·9761	2	1181·9522
37·5	160·00	+34·31	1177·1761	2	2354·3522
Totals				29 $= \Sigma f = n$	7105·924 $= \Sigma(fd_y^2)$

Blakeman's Criterion for Linearity of Regression.—When mere inspection of the means of the arrays (the crosses and circles in fig. 164) affords no reliable information regarding the nature of the regression, the problem can be decided by means of Blakeman's Criterion which is to the effect that if $(\eta^2 - r^2)$

is more than 3 times its probable error (p.e.), the regression is curvilinear; if it is considerably less than 3 p.e. the regression is rectilinear.

The p.e. of $(\eta^2 - r^2)$ is

$$\frac{2 \times 0{\cdot}6745}{\sqrt{n}}\sqrt{\eta^2 - r^2}\sqrt{(1-\eta^2)^2 - (1-r^2)^2 + 1}$$

(n = number of observations).

As η and r generally differ very little from each other, $\sqrt{(1-\eta^2)^2 - (1-r^2)^2 + 1}$ does not differ greatly from unity. Also $2 \times 0{\cdot}6745$ may be taken roughly as 4/3. Hence, *regression is rectilinear or curvilinear* according as $(\eta^2 - r^2)$ is less or greater than $\frac{4}{\sqrt{n}}\sqrt{\eta^2 - r^2}$, *i.e. according as* $\sqrt{n}\sqrt{\eta^2 - r^2}$ *is less or greater than* 4.

Thus, in the case of overcrowding and infant mortality this expression is equal to $\sqrt{29}\sqrt{0{\cdot}74^2 - 0{\cdot}64^2} = 2$, showing that the regression is rectilinear.

On the other hand, in the case of 454 observations of weights and sitting heights of embryos, when η and r were found to be 0·98 and 0·94 respectively,

$$\sqrt{n}\sqrt{\eta^2 - r^2} = \sqrt{454}\sqrt{0{\cdot}98^2 - 0{\cdot}94^2} = 5{\cdot}9,$$

showing that the regression is curvilinear. This confirms what has already been found, that a straight line will not fit the data in the example on p. 426 and fig. 156.

EXAMPLES.

(1) From a table giving the heights and weights of a number of boys the following results were found:—

$$\Sigma xy = 1630{\cdot}49$$
$$\Sigma x^2 = 644{\cdot}77 \quad (x = \text{deviation in height}).$$
$$\Sigma y^2 = 14642{\cdot}06 \quad (y = \text{deviation in weight}).$$

Find the correlation coefficient between height and weight.

$$r = \frac{\Sigma(xy)}{n\sigma_x\sigma_y} = \frac{\Sigma(xy)}{n\sqrt{\frac{\Sigma x^2}{n} \cdot \frac{\Sigma y^2}{n}}}$$

$$= \frac{\Sigma(xy)}{\sqrt{\Sigma x^2 \, . \, \Sigma y^2}}$$

$$= \frac{1630{\cdot}49}{\sqrt{644{\cdot}77 \times 14642{\cdot}06}}$$

$$= \frac{1630{\cdot}49}{3073}$$

$$= 0{\cdot}53.$$

(2) The correlation coefficient for cephalic index between either parent and child should be 0·33, but in the case of American Indians (in whom the family relations are somewhat loose) the coefficient was found to be 0·33 between mother and child and only 0·14 between the mother's husband and child. Find the proportion of children not due to the reputed father. (Udny Yule.)

Let the total number of pairs of observations between reputed father and child $= n$, with $r = 0{\cdot}14$. Let n_1 of these pairs be in the true relationship of father and child, with $r_1 = 0{\cdot}33$, and n_2 pairs be those for which there is no blood relationship, with $r_2 = 0$.

Then assuming that the mean and standard deviations of the cephalic index are the same in both sets of observations, we have (since $n = n_1 + n_2$)

$$r = \frac{\Sigma xy}{(n_1+n_2)\sigma_x\sigma_y} = 0{\cdot}14$$

and
$$r_1 = \frac{\Sigma xy}{n_1\sigma_x\sigma_y} = 0{\cdot}33$$

$$\therefore \quad \frac{n_1}{n_1+n_2} = \frac{r}{r_1} = \frac{0{\cdot}14}{0{\cdot}33}$$

$$\therefore \quad 1 - \frac{n_1}{n_1+n_2} = 1 - \frac{0{\cdot}14}{0{\cdot}33}$$

i.e.
$$\frac{n_2}{n_1+n_2} = \frac{0{\cdot}33 - 0{\cdot}14}{0{\cdot}33}$$
$$= 0{\cdot}58.$$

$\therefore$ Proportion of children not due to reputed father = 58 per cent.

(3) From 86 observations on the accuracy as well as speed of addition, r and η were found to be 0·14 and 0·29 respectively. Find the nature of the regression.

[*Answer*, $\sqrt{86}\sqrt{0{\cdot}29^2 - 0{\cdot}14^2} = 2{\cdot}33$. $\therefore$ Regression is rectilinear.]

Partial or Net Correlation.—The correlation so far considered is called **gross** or **multiple correlation,** since it measures the degree of association between one variable y and another variable x without taking into consideration the possible association between x and other variables which have an influence on y. Thus the daily amount of heat (h) produced by the human body is correlated with stature (s) (r_{sh} being 0·6149); but stature is closely correlated with age, weight, body surface, etc., each of which has an influence on heat production; so that to find the true relationship between s and h we would have to eliminate the influence of the concomitants of s that we have mentioned upon h. Such a correlation coefficient between two variables obtained after eliminating the effect of the various concomitants of one variable upon the other is called a **partial** or **net correlation coefficient.**

Let the n variables under consideration (say, stature, heat production, age, weight, body surface, etc.) be $x_1, x_2, x_3, \ldots x_n$;

then the gross coefficient between any two of them, such as between x_1 and x_2, or x_2 and x_3, etc., is denoted symbolically by r_{12} ("*r* one two"), or r_{23} ("two three"), etc. The partial coefficient between, say, x_1 and x_2 when the effect of *one* of the other variables, say, x_3, has been eliminated is denoted symbolically by $r_{12\cdot3}$ ("*r* one two point three")—the subscript after the dot indicating the variable whose influence has been eliminated—and is called a coefficient of the **first order.** Similarly $r_{12\cdot34}$, $r_{23\cdot14}$, etc., which denote the partial coefficients obtained after the elimination of the effects of *two* of the other variables, are called coefficients of the **second order,** and generally, $r_{12\cdot345\ldots n}$, in which the effects of all the remaining $(n-2)$ variables have been eliminated is called a coefficient of the $(\mathbf{n-2})^{\text{th}}$ order.

The chemist, physicist or even physiologist who, with an unlimited supply of material at his disposal, wishes to investigate the net correlationship between any two variables, can always choose his material or arrange his experiments in such a way that the only variables present are those between which he seeks to ascertain the degree of correlation. Thus, he can investigate the effect of pressure upon the volume of a gas *under constant temperature*, or the response of animals or plants to stimuli *under certain constant conditions.* In many biological or sociological investigations, however, such a procedure is impossible or impracticable, because in any given random sample of the population the number of individuals who differ in one particular character only, such as stature, without at the same time differing in the usual accompaniments of stature, viz. weight, age, conformation of body, etc., is so small as to render the result of any computation of the net correlationship between that variable (stature) and, say, heat production, for constant age, weight, etc., of no statistical significance. This difficulty has been overcome mathematically, since—by methods analogous to those of partial differentiation—formulæ have been established which express the relationship between net or partial and gross correlation coefficients, so that from a knowledge of the latter the former may be computed.

Formulæ for Partial Coefficients.—It can be shown that for

First Order Coefficient $r_{12\cdot3} = \dfrac{r_{12} - r_{13}r_{23}}{\sqrt{1-r_{13}^2}\sqrt{1-r_{23}^2}}$, so that, knowing

the three gross coefficients r_{12}, r_{13}, r_{23}, the value of $r_{12\cdot3}$ (as well as $r_{13\cdot2}$ and $r_{23\cdot1}$) may at once be computed.

Second Order Coefficient $r_{12\cdot34} = \dfrac{r_{12\cdot3} - r_{14\cdot3}r_{24\cdot3}}{\sqrt{1-r_{14\cdot3}^2}\sqrt{1-r_{24\cdot3}^2}}$, in which

$r_{12\cdot3}$, $r_{14\cdot3}$ and $r_{24\cdot3}$ have to be evaluated by means of the fundamental formula for the first order coefficient.

$(n-2)$th Order Coefficient

$$r_{12\cdot345\ \ldots\ n} = \frac{r_{12\cdot345\ \ldots\ (n-1)} - r_{n\cdot345\ \ldots\ (n-1)}r_{2n\cdot345\ \ldots\ (n-1)}}{\sqrt{1-r^2_{n\cdot345\ \ldots\ (n-1)}}\sqrt{1-r^2_{2n\cdot345\ \ldots\ (n-1)}}}$$

which is the most general expression for the net correlation coefficient between any two out of n variables when the other $(n-2)$ variables are kept constant.

Note.—The computation of an $(n-2)$th order coefficient needs an evaluation of $n(n-1)/2$ zero order coefficients. Thus a first order coefficient (*i.e.* when $n = 3$) needs, as we have seen, the evaluation of three zero order coefficients $(3\times2/2 = 3)$. A second order coefficient $(n = 4)$, entails the computation of six zero order coefficients $(4\times3/2 = 6)$, and so on. As n increases, the number of zero order coefficients needed in the computation of the $(n-2)$th order coefficient therefore grows very rapidly. The tedious arithmetical labour is, however, greatly reduced by the use of special tables giving the values of $\sqrt{1-r^2}$ for different values of r. In the absence of such tables the work is considerably shortened by using the identity $\cos x = \sqrt{1-\sin^2 x}$ (see p. 34). Thus if $r = 0{\cdot}5725 = \sin 34°\ 55'$, then $\sqrt{1-r^2} = \cos 34°\ 55' = 0{\cdot}8200$.

EXAMPLE.

(1) Out of a total of 136 observations the gross correlation coefficients between height (x_1) and basal metabolism (x_2), between height and weight (x_3) and between weight and metabolism were found to be $r_{12} = 0{\cdot}6149$, $r_{13} = 0{\cdot}5725$ and $r_{32} = 0{\cdot}7960$. Evaluate $r_{12\cdot3}$ and estimate its significance.

$$r_{12\cdot3} = \frac{r_{12}-r_{13}r_{23}}{\sqrt{1-r^2_{13}}\sqrt{1-r^2_{23}}} = \frac{0{\cdot}6149 - 0{\cdot}5725\times0{\cdot}7960}{\sqrt{1-0{\cdot}5725^2}\sqrt{1-0{\cdot}7960^2}} = 0{\cdot}321.$$

The probable error $= \dfrac{0{\cdot}6745(1-0{\cdot}321^2)}{\sqrt{136}}$ (see p. 456) $= \pm0{\cdot}0519$, *i.e.* about $\frac{1}{6}\ r_{12\cdot3}$.

There is therefore a definite correlation between height and metabolism for constant weight, although the fact that the gross coefficient has been nearly halved shows that the main correlationship between height and metabolism is due to that between height and weight.

EXERCISES.

(1) Out of 96 observations on 94 new-born infants (in whom age, x_4, is of course constant) it was found that $r_{12} = 0{\cdot}6848\pm0{\cdot}0369$, $r_{13} = 0{\cdot}8209 \pm0{\cdot}0227$ and $r_{32} = 0{\cdot}7833\pm0{\cdot}0269$ (x_1 = length, x_2 = basal metabolic rate and x_3 = weight). Find $r_{12\cdot34}$. [*Answer*, $0{\cdot}1178\pm0{\cdot}0686$.]

(2) The 96 observations in the previous exercise concerned 51 male infants, in whom r_{12} was $0{\cdot}6191\pm0{\cdot}0582$ and r_{32} was $0{\cdot}7520\pm0{\cdot}0411$, and 43 females for whom $r_{12} = 0{\cdot}7426\pm0{\cdot}0461$ and $r_{32} = 0{\cdot}8081\pm0{\cdot}0357$. Is one justified in saying that there is a correlation between sex and metabolism?

[*Answer*, No. Difference in $r_{12} = 0{\cdot}1235\pm0{\cdot}0741$; difference in $r_{32} = 0{\cdot}0561\pm0{\cdot}0544$.]

(3) If x_1 = buffer action of urine, and x_2, x_3 and x_4 = urinary phosphorus, sulphur and calcium respectively, then $r_{12} = 0{\cdot}885$, $r_{13} = 0{\cdot}335$, $r_{14} = 0{\cdot}772$, $r_{23} = 0{\cdot}634$, $r_{24} = 0{\cdot}555$ and $r_{34} = 0{\cdot}391$. Calculate the values of $r_{23\cdot14}$ and $r_{24\cdot13}$. [*Answer*, $r_{23\cdot14} = +0{\cdot}985$; $r_{24\cdot13} = -0{\cdot}909$.]

Correlation and Causation.—It cannot be too strongly emphasised that correlation is not necessarily synonymous with causation. Thus while there is a tendency for similarity of heights or eye-colour between husband and wife, it is clear that such correlation can only be due to *selective* mating and not in any way to the mating *per se*. On the other hand, between parent and child such a correlation for physical characters must entirely be of cause and effect. The experimental scientist can usually arrange his experiments in such a way as to discover the existence of causal relationship between two associated phenomena.

The human biologist or sociologist, who cannot resort to the experimental method, must rely upon logic and common sense for a solution of the problem.

Table I.—LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10

Table I.—LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5

Table I.—LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Table II.—FUNCTIONS OF ANGLES.

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0°	0	000	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1929	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
								Radians.	Degrees.
			Cosine.	Co tangent.	Tangent.	Sine.	Chord.	Angle.	

Table III.—PROBABILITY INTEGRALS.

The following is a short probability integral table giving values of $\frac{1}{\sqrt{2\pi}}\int_0^\xi e^{-\xi^2/2}d\xi$, *i.e.* areas of portions of normal frequency curve bounded by the ordinates at $\frac{x}{\sigma} = 0$ and $\frac{x}{\sigma} = \xi$ (viz. areas of ONQP (*i.e.* A/2) in fig. 155 for different values of $\frac{x}{\sigma}$), as well as the heights of the ordinates $y = \frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}$ at different abscissal points.

$\xi = \frac{x}{\sigma}$	$\frac{A}{2} = \frac{1}{\sqrt{2\pi}}\int_0^\xi e^{-\xi^2/2}d\xi$	$y = \frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}$	ξ	$\frac{A}{2}$	y
0·00	0·0000	0·3989	0·90	0·3159	0·2661
0·10	0·0398	0·3970	0·95	0·3289	0·2541
0·20	0·0793	0·3910	1·00	0·3413	0·2420
0·30	0·1179	0·3814	1·05	0·3531	0·2299
0·40	0·1554	0·3683	1·10	0·3643	0·2179
0·45	0·1736	0·3605	1·5	0·4332	0·1295
0·50	0·1915	0·3521	2·0	0·4772	0·0540
0·55	0·2088	0·3429	2·5	0·4938	0·0175
0·60	0·2257	0·3332	3·0	0·4987	0·0044
0·65	0·2422	0·3230	3·5	0·4998	0·0009
0·70	0·2580	0·3123			
0·71	0·2611	0·3101	4·0	$0·5 - 3·2 \times 10^{-5}$	0·0001
0·72	0·2642	0·3079	4·5	$0·5 - 3 \times 10^{-6}$	$1·6 \times 10^{-5}$
0·73	0·2673	0·3056	5·0	$0·5 - 3 \times 10^{-7}$	$1·5 \times 10^{-5}$
0·74	0·2703	0·3034	5·5	$0·5 - 2 \times 10^{-8}$	1×10^{-6}
0·75	0·2734	0·3011	6·0	$0·5 - 1 \times 10^{-9}$	6×10^{-8}
0·76	0·2764	0·2989			
0·77	0·2794	0·2966			
0·78	0·2823	0·2943			
0·79	0·2852	0·2920			
0·80	0·2881	0·2897			
0·85	0·3023	0·2780			

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